UNIVERSIDAD ANDRÉS BELLO

DOCTORAL THESIS

Kounterterms in Einstein Gravity and in Higher-Derivative Theories of Gravity

---

Author:
Georgios Anastasiou

Supervisor:
Dr. Rodrigo Olea

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in the

Grupo de Gravitación y Altas Energías
Departamento de Ciencias Físicas

June 1st, 2018

Declaration of Authorship

I, Georgios Anastasiou, declare that this thesis titled, "Kounterterms in Einstein Gravity and in Higher-Derivative Theories of Gravity" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:

"I don't want to believe, I want to know"

Carl Sagan

UNIVERSIDAD ANDRÉS BELLO

Abstract

Facultad de Ciencias Exactas
Departamento de Ciencias Físicas

Doctor of Philosophy

Kounterterms in Einstein Gravity and in Higher-Derivative Theories of Gravity

by Georgios Anastasiou

The aim of this thesis is two-fold. It consists mainly in gaining better intuition on the mechanism of cancellation of IR divergences in Einstein gravity with negative cosmological constant, using a new regularization scheme, namely the Kounterterms. Exploiting its properties and its connection with topological invariance, we gain insight both in Conformal and Critical Gravity, which are higher-derivative theories defined in 4D.

The Kounterterms stand as an alternative regularization scheme to Holographic Renormalization and corresponds to the addition of surface terms that depend on intrinsic and extrinsic quantities of the boundary on top of the Einstein-Hilbert action, rendering the action evaluated on Anti de-Sitter spacetimes, finite. Unlike the counterterms in the standard Holographic Renormalization, they admit a closed form in any dimension. We compare two regularization schemes in Asymptotically Conformally Flat spaces, and conclude their equivalence for terms up to cubic order in the curvature $\mathcal{O}\left(\mathcal{R}^3\right)$.

Because of the connection between the Kounterterms and topological invariants in even-dimensional spacetimes, the renormalized Einstein-AdS action admits a closed form, expressed as a polynomial of the on-shell Weyl tensor. This feature is crucial when dealing with higher-curvature gravity theories with an Einstein branch of solutions. In Conformal Gravity it allows us to prove explicitly the equivalence between Einstein and Conformal Gravity, previously stated by Maldacena.

In Critical Gravity, using the Noether-Wald method we prove, in a non-perturbative way, the trivial character of the Einstein modes of the theory. Indeed, the only non-trivial information is coming from the non-Einstein modes represented by the Bach tensor. Based on this result, we introduce a new series of counterterms, which depend explicitly on the extrinsic curvature and its derivatives, and provide a shortcut in the derivation of the holographic correlations functions living in the boundary Conformal Field Theory.

Acknowledgements

Firstly, I would like to express my sincere gratitude to my advisor, Rodrigo Olea, for the guidance during my PhD, his continued support and for the things that he taught me all these years. He introduced me in the world of the AdS/CFT correspondence and our discussions has been proven valuable lessons for my future in many aspects of my life. I am thankful for all these numerous hours in the office, where you taught me that hard work is the only path to success.

I'm grateful to Olivera Miskovic for reading carefully this thesis, pointing out her detailed comments and corrections. Her enthusiasm for Physics was a true inspiration for me. Our discussions helped me clarify many issues and deepen my knowledge in holography and its techniques. I still remember in our latest collaboration working along with Rodrigo in his apartment when an earthquake of magnitude of 8.5 R struck Santiago. Nothing to worry about, that's life in Chile.

In the last year of my PhD I had the honour to meet my good friend and great physicist, Ignacio Araya. We entered hand by hand in the exciting new world of Quantum Information Theory. I couldn't hope for a better officemate and collaborator. There have been countless discussions in our office, in front of a whiteboard or at the "Republicano" drinking a Kross 5.

Special thanks to the very talented young physicists of the High Energy Physics Group of UNAB, David Rivera-Betancour, Gabriel Arenas, Felipe Diaz, Pablo Guilleminot and Javier Moreno, for your friendship, support and discussions. You were the audience of many parts of the PhD thesis. Your comments and criticisms only made it better.

Heartfelt thanks to Cesar Arias for his patience listening to my whining for the defects of the PhD program. I admire your scientific skills and I'm happy that I had the chance to collaborate with you. Thank you for being a good companion in many chelas. I would like to thank Israel Ramirez for our morning coffees, useful discussions and tasty dinners in his house. I am deeply indebted to the rest of the faculty, the postdocs as well as the secretary of the department of Physics.

Many thanks to all my good Chilean friends that they welcomed me in their beautiful country and they made me feel like home, Raul Guerrero, Rene Araneda, Dario Lopez, Sebastian Acevedo, Ignacio Diaz and Wilson Caimanque.

Of course, my eternal gratitude to my family, my parents, my brother and my two sisters, for their support and faith since I was a young child. Thank you for encouraging me to follow my dreams.

Contents

List of Abbreviations

IR	Infrared
CG	Conformal Gravity
CrG	Critical Gravity
AdS	Anti de-Sitter
HR	Holographic Renormalization
EH	Einstein-Hilbert
ACF	Asymptotically Conformally Flat
CFT	Conformal Field Theory
FG	Fefferman-Graham
ALAdS	Asymptotically Locally Anti de-Sitter
UV	Ultraviolet
LCFT	Logarithmic Conformal Field Theory
AAdS	Asymptotically Anti de-Sitter
EOM	Equations of Motion
CS	Chern-Simons
GB	Gauss-Bonnet
SFF	Second Fundamental Form
GR	General Relativity
NMG	New Massive Gravity
TMG	Topologically Massive Gravity
EW	Einstein-Weyl
E	Einstein
NE	non-Einstein
OPE	Operator product expansion
GC	Gauss-Codazzi
RHS	Right-hand side
LHS	Left-hand side
GH	Gibbons-Hawking
TT	Transverse-traceless

Dedicated to my family

Chapter 1

Introduction

The AdS/CFT correspondence constitutes a concrete realization of holography with many applications in various areas of Physics. It expresses the duality between the dynamical fields propagating in the bulk AdS_{d+1} spacetime and the operators of the CFT_d that live in one dimension lower, at the conformal boundary of the AdS spacetime [1]. Assuming a classical gravity theory, namely weakly coupled, and its strongly coupled CFT dual, then the following equivalence between the generating functional of the boundary CFT and the bulk on-shell action is valid:

$$S_{on-shell}\left(\varphi_{(0)}\right) = -W_{CFT}\left(\varphi_{(0)}\right) , \tag{1.1}$$

where W_{CFT} is the quantum effective action of the CFT, $S_{on-shell}$ is the on-shell bulk action and $\varphi_{(0)}$ is the value at the conformal boundary of the bulk propagating field φ [2]. The above relation allows us to compute the correlation functions of the gauge operators living on the boundary, using the on-shell bulk action.

The asymptotic conformal structure of AdS spacetimes induces a series expansion of the bulk fields around the conformal boundary, where $\varphi_{(0)}$ is the leading contribution and plays the role of the source of the dual operator. Solving the equations of motion order by order, one determines several terms of the asymptotic expansion as functions of the source. Thus, defining Dirichlet boundary conditions for the source $\varphi_{(0)}$, one can reconstruct the bulk field partially up to the deep interior.

A concrete realization of this construction is the FG expansion of the bulk metric $\mathcal{G}_{ij}$ [3]. In this case the conformal boundary metric $g_{(0)ij}$ is the source that couples to the boundary stress-energy tensor $\langle T_{ij} \rangle$, which is the dual operator. The presence of a pole of second order in this class of spacetimes, called ALAdS spacetimes, induces a conformal structure in asymptotic infinity that introduces infinities at the action. In the context of AdS/CFT correspondence, ALAdS spacetimes are of wide interest, as the form of the boundary metric, which corresponds to the CFT background metric, is totally unconstrained.

As a consequence, the on-shell action of the gravitational action diverges, reflecting the infinity of the bulk volume. In the context of holography, the bulk divergences are seen as the UV divergences arising in the dual field theory. This feature is actually an aspect of the duality between the radial coordinate in the bulk and the energy scale in the boundary.

In order to cancel the divergences, requiring the action to be well-defined, a series of Dirichlet surface terms has to be added, namely the counterterms. This means that the counterterms depend only on intrinsic quantities of the codimension-1 hypersurface at constant radius,I.e., the metric of the hypersurface, the intrinsic curvature and derivatives of the curvature. Based on these considerations the HR was introduced in [4]–[6], as a systematic regularization scheme of the bulk action.

Even though the HR provides a systematic way to render the action finite, the complexity of the counterterms grows as we go to higher dimensions. Moreover, the Dirichlet variational problem for the full boundary metric h_{ij} is not well defined, as it diverges at spatial infinity. A consistent choice is the fixing of the background metric of the CFT, namely $g_{(0)ij}$. These two features motivated the discussion for alternative regularization schemes.

In [7] it was proposed a different prescription for the cancellation of the divergences. This new scheme is based on the addition of a unique surface term that depends both on intrinsic $(\mathcal{R})$ and extrinsic (K) curvatures with a fixed coupling constant (c_d). This new proposal, which is characterized by its compactness and its profound connection to geometry, is called Kounterterms. Its general form for AdS_{d+1} reads

$$\tilde{I}_{ren} = I_{EH} + c_d \int\limits_{\partial M} d^d x \sqrt{-h} B_d \left(h, K, \mathcal{R}\right) . \tag{1.2}$$

For even dimensions the surface term arises as the boundary correction to the Euler theorem for non-compact manifolds with a boundary. As a consequence there is an underlying relation between the regularization of the action and the addition of topological invariants of the Euler class. This concept is called Topological Regularization.

Despite the absence of topological invariants in odd-dimensional manifolds, an analogous regularizing scheme had been introduced, generalizing the concept of Kounterterms [8]. In this case the geometrical interpretation is different, as the term arises as a boundary correction to the Transgression form of the AdS group.

In chapter 2, we compare the two regularizing schemes, namely the Kounterterms and the standard counterterm series $\mathcal{L}_{ct}(h)$. In order for the two formulas to coincide, we add and subtract the Gibbons-Hawking term in Eq. (1.2). The main obstacle, is the transition from the extrinsic counterterms to intrinsic ones. The problem is circumvented by considering that the extrinsic curvature K_{ij} can be expanded asymptotically as a function of intrinsic quantities of the conformal boundary.

It is shown their full accordance up to eighth order derivative terms, which arise in the boundary of ten and eleven dimensions. Based on this equivalence, we show that the off-shell form of the standard counterterm can be written in a compact form and a general recursive formula for $(2p)$-order counterterm is proposed, common for both even and odd-dimensional manifolds.

In chapter 3 of this thesis, we provide an explicit proof of the equivalence between Einstein and CG based on Topological Regularization in 4D [9]. From the derivation of the field equations we realize that Einstein spacetimes constitute a specific branch in the broader class of solutions of the theory. Based on this, we decompose the curvature into an Einstein and a NE part, which can be extended at the level of the action and the variation of it, respectively. It is shown that the vanishing of the Bach tensor is a sufficient condition in order to recover Einstein gravity with a negative cosmological constant.

Finally, a direct consequence of this result is presented in chapter 4, in the context of CrG. It is explained the triviality of the theory when the Einstein branch is considered [10]. We show that the only non-trivial contribution is coming from a function quadratic in the Bach tensor [11]. The presence of NE modes in the theory demands a modified asymptotic behavior that switches on a new source on the boundary theory. Furthermore, the finiteness of the action is analyzed and a new series of counterterms is proposed depending explicitly on the extrinsic curvature and its derivatives. As a final step, the holographic correlation functions are computed and the boundary theory is identified as a LCFT.

Chapter 2

Asymptotic analysis of AdS spacetimes

2.1 Introduction

The holographic principle is based on the conjecture about the duality between a gravity theory living in the ($D = d + 1$)-dimensional bulk manifold and a d-dimensional field theory residing on the boundary. In a more concrete way, the duality states that the bulk propagating fields determine the observables of the boundary field theory. A concrete realization of the holographic principle is given by the AdS/CFT correspondence, where on the one side we have fields on a gravitational background equipped with a negative cosmological constant ($\Lambda < 0$) and on the other side, gauge invariant operators in the boundary CFT [1]. This one to one correspondence between the fields is a direct consequence of the identification of the partition functions of the two theories, which reads

$$Z_{SUGRA}\left(\varphi_{(0)}\right) = Z_{CFT}\left(\varphi_{(0)}\right) , \tag{2.1}$$

where $Z_{SUGRA}\left(\varphi_0\right)$ is the partition function in the low energy regime of string theory (Supergravity), expressed as a function of the boundary values of the bulk fields, and Z_{CFT} is the partition function of the strongly coupled CFT [2]. In the saddle-point approximation, $Z_{SUGRA} \simeq e^{-I_{on-shell}}$ and the Eq. (2.1) can equivalently be written as

$$I_{on-shell}\left(\varphi_{(0)}\right) = -W_{CFT}\left(\varphi_{(0)}\right) , \tag{2.2}$$

implying the equivalence between the on-shell gravitational action and the CFT generating functional. The generating functional determines the n-point functions of the theory. In particular, one-point function reads

$$\langle \mathcal{O}\left(x\right) \rangle = -\frac{\delta W_{CFT}\left(\varphi_{(0)}\left(x\right)\right)}{\delta \varphi_{(0)}\left(x\right)} , \tag{2.3}$$

where $\varphi_{(0)}$ is the value of the the bulk field (scalar, gauge field or the gravitational field) at the conformal boundary. Here $\varphi_{(0)}$ plays also the role of the source that couples to the dual operator $\mathcal{O}\left(x\right)$.

Following the AdS/CFT dictionary, the Eq. (2.3) reads

$$\langle \mathcal{O}\left(x\right) \rangle = \frac{\delta I_{on-shell}\left(\varphi_{(0)}\right)}{\delta \varphi_{(0)}} . \tag{2.4}$$

Hence, one determines the operators and the correlations functions of the strongly coupled CFT by calculating the on-shell action of the weakly coupled gravitational counterpart.

In the analysis below, we are focusing on gravity. In this case, the dual to the bulk metric $\mathcal{G}_{\mu\nu}$ is the stress-energy tensor $\left\langle T_{ij}\left(x\right)\right\rangle$ of the CFT_d, which is written as 1-point function

$$\left\langle T_{ij}\left(x\right)\right\rangle = \frac{2}{\sqrt{-g_{(0)}}} \frac{\delta I_{on-shell}\left(g_{(0)}\right)}{\delta g^{ij}_{(0)}\left(x\right)} . \tag{2.5}$$

Here, $g_{(0)ij}$ is the background metric of the boundary CFT which is also the source that couples to the stress-energy tensor. In the gravitational context, it corresponds to the value of the bulk metric $\mathcal{G}_{ij}$ when evaluated at the boundary and defines the Dirichlet boundary conditions of the theory. It is evident that the study of the asymptotic structure of AdS spacetimes is crucial for the understanding of holography.

Following the Refs.[12], [13], the class of spacetimes which are asymptotically exact AdS, they are called AAdS spacetimes. This means that the bulk Riemann tensor in the asymptotic region obtains the form

$$R^{\alpha\beta}_{\mu\nu} = -\frac{1}{\ell^2}\delta^{[\alpha\beta]}_{[\mu\nu]} \tag{2.6}$$

where $\delta^{[\alpha\beta]}_{[\mu\nu]} = \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}$ is the generalized antisymmetric delta. The following metric

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + dx_\mu dx^\mu\right) \tag{2.7}$$

represents AdS in Poincare coordinates and satisfies the condition (2.6). We realize that at the AdS boundary, which is located at $z = 0$, the spacetime (2.7) is singular due to the second order pole. This means that no boundary metric is induced from the bulk. In order to overcome the boundary singularity we employ Penrose's conformal compactification technique. Considering a function $z(x)$, which is positive in the bulk and has a first order pole at the boundary, one defines the metric

$$\tilde{\mathcal{G}} = z^2 \mathcal{G}\,, \tag{2.8}$$

which is conformally related to $\mathcal{G}_{ij}$ and regular at the whole space. This means that the new metric can be smoothly extended to the boundary without the presence of any singularities. Indeed, the boundary metric, given by

$$g_{(0)} = z^2 \mathcal{G}_{z=0}\,, \tag{2.9}$$

is regular but can only be determined up to the choice of the function $z(x)$. Actually, every defining function of the same conformal class with $z(x)$ is valid. As a consequence, the boundary metric g_0 is determined up to a conformal factor. Hence, the bulk metric $\mathcal{G}$ induces a conformal structure at the boundary. In the pure AdS case (2.7), the induced metric at the boundary is Minkowski and belongs to the same conformal class with the Einstein static universe metric which is topologically $R \times S^{d-1}$. This is a crucial feature of the AAdS spacetimes.

The above considerations were generalized for a generic (pseudo) Riemannian metric G with a second order pole, similar to the one seen in the pure AdS case. If there is a defining function z which is positive definite in the bulk and satisfies the conditions $z(\partial M) = 0, dz(\partial M) \neq 0$ at the boundary of a manifold M, then G is a conformally compact metric as $\tilde{G} = z^2 G$ is smooth with a regular boundary metric $g_{(0)} = z^2 G_{|_{\partial M}}$. The choice of the defining function is not unique, leading to many different compactifications of the bulk metric. Therefore, the bulk metric induces a conformal structure at the boundary, located in $z = 0$, defined by the conformal class of the boundary metric $[g_0]$.

When G is a solution of the Einstein equations then it represents a conformally compact Einstein manifold whose Riemann curvature reads

$$R^{\alpha\beta}_{\mu\nu} = -\frac{1}{\ell^2}\delta^{[\alpha\beta]}_{[\mu\nu]} + \mathcal{O}\left(z^{-3}\right)\,. \tag{2.10}$$

We realize that the Riemann tensor of the conformally compact Einstein manifolds approaches asymptotically the pure AdS form given in (2.6), justifying the name AlAdS spaces. The key point here is that there is no restriction on the conformal structure and the topology of the boundary, unlike the AAdS case where the boundary metric is conformal to $R \times S^{d-1}$. As a consequence, the AlAdS spacetimes correspond to more generic asymptotics than AAdS spaces, in accordance with the non-restrictive form of the source of the dual CFT. In [3], it was introduced a coordinate system representing the conformal structure induced at infinity. More specifically, considering that the defining function $z_{(x)}$ plays the roles of the radial coordinate in a Gauss-normal frame emanating from the boundary, the bulk metric reads

$$ds^2 = \frac{1}{z^2}\left(dz^2 + g_{ij}\left(z, x\right)dx^i dx^j\right)\,. \tag{2.11}$$

As it was shown in (2.8), the metric $g_{ij}(z,x)$ is regular at the asymptotic limit $z \to 0$. Thus endows a Taylor-like expansion around the conformal boundary $z = 0$, which casts the form

$$\begin{aligned} g_{ij}(z,x) &= g_{(0)ij}(x) + z g_{(1)ij}(x) + z^2 g_{(2)ij}(x) + \ldots \\ &\ldots + z^d g_{(d)ij}(x) + z^d h_{(d)ij} \log z^2 \,. \end{aligned} \tag{2.12}$$

In Einstein gravity, all the odd-power coefficients vanish, when solving Einstein's equations in this frame. It is convenient to rewrite the FG expansion considering the coordinate transformation $\rho = z^2$. In this case the bulk metric in FG coordinates obtains the form

$$ds^2 = \frac{\ell^2}{4\rho^2} d\rho^2 + h_{ij}(x,\rho) dx^i dx^j \,, \tag{2.13}$$

where $h_{ij}(x,\rho) = \frac{1}{\rho} g_{ij}(x,\rho)$ is the induced metric on the $\rho = const$ hypersurface and $g_{ij}(x,\rho)$ expresses the series expansion around the boundary $\rho = 0$

$$\begin{aligned} g_{ij}(\rho,x) &= g_{(0)ij}(x) + \rho g_{(2)ij}(x) + \rho^2 g_{(4)ij}(x) + \rho^3 g_{(6)ij}(x) + \ldots \\ &\ldots + \rho^{\frac{d}{2}} g_{(d)ij}(x) + \rho^{\frac{d}{2}} h_{(d)ij} \log \rho \,. \end{aligned} \tag{2.14}$$

The logarithmic term arises only in the case of an even-dimensional boundary, and its coefficient determines the variation of the conformal anomaly with respect to the metric. Considering that the indices are raised and lowered by the metric $g_{(0)ij}$, the contravariant form of the expansion (2.14) now reads

$$\begin{aligned} g^{ij}(\rho,x) = g^{ij}_{(0)} - \rho g^{ij}_{(2)} + \rho^2 \left[g^{ik}_{(2)} g^{j}_{(2)k} - g^{ij}_{(4)} \right] - \\ -\rho^3 \left[g^{i}_{(2)k} g^{k}_{(2)m} g^{mj}_{(2)} + g^{ij}_{(6)} - 2 g^{i}_{(2)k} g^{kj}_{(4)} \right] + \mathcal{O}\left(\rho^4\right) \,. \end{aligned} \tag{2.15}$$

Actually, the fixing of the leading order term defines the Dirichlet boundary conditions in the gravitational context. Namely, given $g_{(0)ij}(x)$, one determines partially the bulk metric up to the deep interior, but not to all orders, by solving the Einstein equations order by order in this frame. This is a consequence of the fact that additional boundary data is needed in order to be able to compute the divergenceless and traceless part of $g_{(d)ij}$. In the following section, we will compute the first terms of the FG expansion and show that they are functions of the intrinsic curvature of $g_{(0)}$.

2.2 Einstein equations in the FG frame and the asymptotic fall-off of the Weyl tensor

2.2.1 Determining the FG coefficients

The EOM of Einstein-AdS gravity, meaning the EH action equipped with a negative cosmological constant, in a $(d+1)$-dimensional manifold M is given by

$$R^{\mu}_{\nu} - \frac{1}{2}R\delta^{\mu}_{\nu} - \frac{d\,(d-1)}{\ell^2}\delta^{\mu}_{\nu} = 0\,. \tag{2.16}$$

Taking the trace of the EOM, one gets that

$$R^{\mu}_{\nu} = -\frac{d}{\ell^2}\delta^{\mu}_{\nu}\,, \tag{2.17}$$

which defines the class of Einstein spaces. The Weyl tensor, on the other hand, is defined by

$$W^{\alpha\beta}_{\mu\nu} = R^{\alpha\beta}_{\mu\nu} - S^{\alpha}_{\mu}\delta^{\beta}_{\nu} + S^{\beta}_{\mu}\delta^{\alpha}_{\nu} + S^{\alpha}_{\nu}\delta^{\beta}_{\mu} - S^{\beta}_{\nu}\delta^{\alpha}_{\mu}\,, \tag{2.18}$$

where

$$S^{\mu}_{\nu} = \frac{1}{d-1}\left(R^{\mu}_{\nu} - \frac{1}{2d}R\delta^{\mu}_{\nu}\right)\,, \tag{2.19}$$

is the Schouten tensor of the bulk. The Weyl tensor is traceless (W^{a}_{bad}) and is subject to the properties of the Riemann tensor, namely

$$\begin{gathered} W_{abcd} = -W_{bacd} = -W_{abdc} \\ W_{abcd} = W_{cdab} \\ W_{abcd} + W_{acdb} + W_{adbc} = 0\,. \end{gathered}$$

When evaluated on Einstein spacetimes, the Schouten and the Weyl tensors obtain the form

$$S^{\mu}_{\nu} = -\frac{1}{2\ell^2}\delta^{\mu}_{\nu}\,, \tag{2.20}$$

$$W^{\alpha\beta}_{\mu\nu} = R^{\alpha\beta}_{\mu\nu} + \frac{1}{\ell^2}\delta^{[\alpha\beta]}_{[\mu\nu]}\,. \tag{2.21}$$

Considering the tracelessness of the on-shell Weyl tensor, one recovers the Einstein's EOM, because

$$W^{\mu\alpha}_{\nu\alpha} = R^{\mu}_{\nu} + \frac{d}{\ell^2}\delta^{\mu}_{\nu} = 0\,. \tag{2.22}$$

In what follows the Greek indices indicate spacetime indices, while the Latin ones indicate boundary indices. Working in Gauss-normal coordinates for the radial foliation of the bulk metric

$$ds^2 = N^2(\rho)\, d\rho^2 + h_{ij}(x,\rho)\, dx^i dx^j \,, \tag{2.23}$$

where N is the lapse function, the extrinsic curvature is defined by

$$K_{ij} = -\frac{1}{2N}\partial_\rho h_{ij}(x,\rho) \,. \tag{2.24}$$

In general, the extrinsic curvature expresses the embedding of a submanifold in a manifold and depends on the particular foliation. In our case, the Eq.(2.24) represents the embedding of the constant radius hypersurfaces with induced metric h_{ij} into the bulk metric. Furthermore, the non vanishing components of the Christoffel symbols are given by

$$\begin{aligned} \Gamma^{\rho}_{\rho\rho} = \frac{\dot{N}}{N}\,, \Gamma^{\rho}_{ij} = \frac{1}{N}K_{ij}\,, \\ \Gamma^{i}_{\rho j} = -NK^{i}_{j}\,, \Gamma^{i}_{jk} = \Gamma^{i}_{jk}(h) \,, \end{aligned} \tag{2.25}$$

where the dot indicates the radial derivative. The Christoffel symbols of the radial foliation will be extremely useful later, in the derivation of the field equations and the surface terms of the CG. When the extrinsic curvature (2.24) is expressed in FG coordinates, then it adopts the following expansion

$$\begin{aligned} K^i_j &= K^i_{(0)j} + \rho K^i_{(2)j} + \rho^2 K^i_{(4)j} + \rho^3 K^i_{(6)j} \\ &= \frac{1}{\ell}\delta^i_j - \frac{\rho}{\ell} g^i_{(2)j} + \frac{\rho^2}{\ell}\left(g^i_{(2)k} g^k_{(2)j} - 2g^i_{(4)j}\right) - \\ &- \frac{\rho^3}{\ell}\left(g^i_{(2)k} g^k_{(2)m} g^m_{(2)j} + 3g^i_{(6)j} - 3g^i_{(2)k} g^k_{(4)j}\right) + \mathcal{O}\left(\rho^4\right) \,. \end{aligned} \tag{2.26}$$

For a general radial foliation, the spacetime Weyl tensor is decomposed in three independent tensors $(W^{ij}_{kl}, E^i_j, W^{ij}_k)$. Once the GC relations [14], for the radial foliations, are applied in Eq.(2.21), one gets for the different Weyl components

$$
\begin{aligned}
W^{ij}_{kl} &= \mathcal{R}^{ij}_{kl} - K^i_k K^j_l + K^i_l K^j_k + \frac{1}{\ell^2}\delta^{ij}_{kl}\,, &(2.27)\\
E^i_j &= W^{i\rho}_{j\rho} = \frac{1}{N}\dot{K}^i_j - K^i_k K^k_j + \frac{1}{\ell^2}\delta^i_j\,, &(2.28)\\
W^i_{jk} &= W^{i\rho}_{jk} = \frac{1}{N}\left(D_j K^i_k - D_k K^i_j\right)\,, &(2.29)\\
W^{ij}_k &= W^{ij}_{k\rho} = -N\left(D^i K^j_k - D^j K^i_k\right)\,. &(2.30)
\end{aligned}
$$

Hence, in this frame, the Einstein's equations become

$$
\begin{aligned}
W^{ik}_{jk} + E^i_j &= 0\,, &(2.31)\\
E^i_i &= 0\,, &(2.32)\\
W^{ij}_j &= 0\,. &(2.33)
\end{aligned}
$$

Substituting the FG expansion in (2.31, 2.32, 2.33), one obtains a series of algebraic equations that determine all the coefficients $g_{(n)ij}$, for $n < d$, plus the trace and the divergence of $g_{(d)ij}$ as functions of intrinsic quantities of the conformal boundary metric $g_{(0)ij}$. In what follows, we will try to determine the first few coefficients and the relations between the traces of the extrinsic curvature terms. The Riemann tensor is expanded in the vicinity of the boundary as

$$
\mathcal{R}^{ij}_{kl}(g) = \mathcal{R}^{ij}_{(0)kl} + \mathcal{R}^{ij}_{(1)kl}\left(g_{(0)}\right) + \mathcal{R}^{ij}_{(2)kl}\left(g_{(0)}\right) + \ldots\,, \tag{2.34}
$$

$$
\mathcal{R}^{ij}_{kl}(h) = \rho\mathcal{R}^{ij}_{kl}(g)\,. \tag{2.35}
$$

Using the definition $\mathcal{R}^{ik}_{jk} = \mathcal{R}^i_j$, we find the Ricci tensor

$$
\mathcal{R}^i_j(h) = \rho\mathcal{R}^i_j(g) = \rho\mathcal{R}^i_{(0)j}\left(g_{(0)}\right) + \rho^2\mathcal{R}^i_{(1)j}\left(g_{(0)}\right) + \rho^3\mathcal{R}^i_{(2)j}\left(g_{(0)}\right) + \ldots\,. \tag{2.36}
$$

Substituting Eqs.(2.27) and (2.28), the first EOM in Eq.(2.31) can equivalently be written as

$$
\mathcal{R}^i_j(h) - KK^i_j + K^i_l K^l_j + \frac{d-1}{\ell^2}\delta^i_j = -E^i_j\,. \tag{2.37}
$$

The RHS of the above equation, is identified as the electric part of the Weyl tensor, previously defined in Eq. (2.28), and represents a symmetric and traceless spatial tensor. This tensor results after projecting two of the indices of the Weyl tensor at the normal (radial) direction. In the FG frame, where the Eq.(2.26) is considered, becomes

$$\begin{aligned} E^i_j &= \frac{2\rho}{\ell}\left(K^i_{(2)j}+2\rho K^i_{(4)j}+3\rho^2 K^i_{(6)j}\right)-\frac{1}{\ell^2}\delta^i_j-\frac{2\rho}{\ell}K^i_{(2)j} \\ &- \frac{\rho^2}{\ell}\left(2K^i_{(4)j}+\ell K^i_{(2)l}K^l_{(2)j}\right)-\frac{2\rho^3}{\ell}\left(K^i_{(6)j}+\ell K^i_{(2)l}K^l_{(4)j}\right)+\frac{1}{\ell^2}\delta^i_j \\ &= \frac{\rho^2}{\ell}\left(2K^i_{(4)j}-\ell K^i_{(2)l}K^l_{(2)j}\right)+\frac{2\rho^3}{\ell}\left(2K^i_{(6)j}-\ell K^i_{(2)l}K^l_{(4)j}\right). \end{aligned} \tag{2.38}$$

Notice that the $\mathcal{O}(\rho)$ term is identically zero.

Inserting the intrinsic and the extrinsic curvatures from Eqs.(2.36) and (2.26), respectively, in the EOM (2.37), leads to the following expression

$$\begin{aligned} &\frac{\rho}{\ell}\left[\ell\mathcal{R}^i_{(0)j}-(d-2)\,K^i_{(2)j}-K_{(2)}\delta^i_j\right]+ \\ +\;&\frac{\rho^2}{\ell}\left[\ell\mathcal{R}^i_{(1)j}-(d-2)\,K^i_{(4)j}-K_{(4)}\delta^i_j+\ell K^i_{(2)l}K^l_{(2)j}-\ell K^i_{(2)j}K_{(2)}\right] \\ +\;&\frac{\rho^3}{\ell}\left[\ell\mathcal{R}^i_{(2)j}-(d-2)\,K^i_{(6)j}-K_{(6)}\delta^i_j-\ell K_{(2)}K^i_{(4)j}-\ell K_{(4)}K^i_{(2)j}+2\ell K^i_{(2)l}K^l_{(4)j}\right]= \\ =\;&-\rho^2 E^i_{(4)j}-\rho^3 E^i_{(6)j}\,. \end{aligned} \tag{2.39}$$

The $E^i_{(n)j}$ are terms that represent the expansion of the electric part of the Weyl tensor. From the Eq. (2.38), the first two terms are identified by

$$E^i_{(4)j}=\frac{1}{\ell}\left(2K^i_{(4)j}-\ell K^i_{(2)l}K^l_{(2)j}\right), \tag{2.40}$$

$$E^i_{(6)j}=\frac{2}{\ell}\left(2K^i_{(6)j}-\ell K^i_{(2)l}K^l_{(4)j}\right). \tag{2.41}$$

Solving the Eq.(2.39) order by order, we determine the coefficients of the FG expansion of the metric. Starting from the $\mathcal{O}(\rho)$ term, we get

$$\ell\mathcal{R}^i_{(0)j}-(d-2)\,K^i_{(2)j}-K_{(2)}\delta^i_j=0\,. \tag{2.42}$$

The trace of this equation implies that

$$K_{(2)}=\frac{\ell}{2(d-1)}\mathcal{R}_{(0)}\,. \tag{2.43}$$

Thus, when inserted into the $\mathcal{O}(\rho)$ EOM, the next to the leading order term of the extrinsic curvature expansion is determined by,

$$K^i_{(2)j}=\frac{\ell}{d-2}\left[\mathcal{R}^i_{(0)j}-\frac{1}{2(d-1)}\mathcal{R}_{(0)}\delta^i_j\right]=\ell\mathcal{S}^i_{(0)j}\,, \tag{2.44}$$

where $\mathcal{S}^{i}_{(0)j}$ is the Schouten tensor of the conformal boundary metric $g_{(0)ij}$. Furthermore, the expansion in Eq.(2.26) allows us to identify the equivalence

$$K^{i}_{(2)j} = -\frac{1}{\ell} g^{i}_{(2)j} \,, \tag{2.45}$$

and determine the next to the leading order term of the metric in the FG frame as

$$g^{i}_{(2)j} = -\ell^{2} \mathcal{S}^{i}_{(0)j} \,. \tag{2.46}$$

Going to the $\mathcal{O}\left(\rho^{2}\right)$ term, the EOM read

$$\mathcal{R}^{i}_{(1)j} - \ell^{-1}\left(d-2\right) K^{i}_{(4)j} - \ell^{-1} K_{(4)} \delta^{i}_{j} + K^{i}_{(2)l} K^{l}_{(2)j} - K^{i}_{(2)j} K_{(2)} = -E^{i}_{(4)j} \,. \tag{2.47}$$

Taking the trace, we get

$$K_{(4)} = \frac{\ell}{2(d-1)} \left(\mathcal{R}_{(1)} - \delta^{[i_1 i_2]}_{[j_1 j_2]} K^{j_1}_{(2)i_1} K^{j_2}_{(2)i_2} \right) \,, \tag{2.48}$$

and substituting it back into the EOM, we find

$$\begin{aligned} K^{i}_{(4)j} = \frac{\ell}{d-4} \Big[& \mathcal{R}^{i}_{(1)j} - K_{(2)} K^{i}_{(2)j} + K^{i}_{(2)l} K^{l}_{(2)j} - \\ & - \frac{1}{2(d-1)} \left(\mathcal{R}_{(1)} - \delta^{[i_1 i_2]}_{[j_1 j_2]} K^{j_1}_{(2)i_1} K^{j_2}_{(2)i_2} \right) \delta^{i}_{j} \Big] \,. \end{aligned} \tag{2.49}$$

From (2.26), $K^{i}_{(4)j}$ is identified as

$$K^{i}_{(4)j} = \frac{1}{\ell} \left[g^{i}_{(2)k} g^{k}_{(2)j} - 2 g^{i}_{(4)j} \right] \,, \tag{2.50}$$

and considering the Eqs.(2.46) and (2.49), the $g_{(4)ij}$ reads

$$\begin{aligned} g^{i}_{(4)j} = \frac{\ell^{2}}{2(d-4)} \Big[& \left(d-5\right) K^{i}_{(2)k} K^{k}_{(2)j} + K^{i}_{(2)j} K_{(2)} - \mathcal{R}^{i}_{(1)j} + \\ & + \frac{1}{2(d-1)} \left(\mathcal{R}_{(1)} - \delta^{[i_1 i_2]}_{[j_1 j_2]} K^{j_1}_{(2)i_1} K^{j_2}_{(2)i_2} \right) \delta^{i}_{j} \Big] \,. \end{aligned} \tag{2.51}$$

Focusing on the $\mathcal{O}\left(\rho^{3}\right)$ order, the EOM have the form

$$\begin{aligned} & \mathcal{R}^{i}_{(2)j} - \ell^{-1}\left(d-2\right) K^{i}_{(6)j} - \ell^{-1} K_{(6)} \delta^{i}_{j} - K_{(2)} K^{i}_{(4)j} - \\ & - K_{(4)} K^{i}_{(2)j} + 2 K^{i}_{(2)l} K^{l}_{(4)j} = -E^{i}_{(6)j} \,. \end{aligned} \tag{2.52}$$

Taking the trace, the RHS piece vanishes, and gives

$$K_{(6)} = \frac{\ell}{2(d-1)} \left(\mathcal{R}_{(2)} - 2\delta^{[i_1 i_2]}_{[j_1 j_2]} K^{j_1}_{(2) i_1} K^{j_2}_{(4) i_2} \right) . \tag{2.53}$$

Putting this result back in the EOM, one gets

$$\begin{aligned} K^i_{(6)j} = \frac{\ell}{d-6} \Big[& \mathcal{R}^i_{(2)j} - K_{(2)} K^i_{(4)j} - K_{(4)} K^i_{(2)j} - \\ & - \frac{1}{2(d-1)} \left(\mathcal{R}_{(2)} - 2\delta^{[i_1 i_2]}_{[j_1 j_2]} K^{j_1}_{(2) i_1} K^{j_2}_{(4) i_2} \right) \delta^i_j \Big] . \end{aligned} \tag{2.54}$$

As a last step, we show that the tracelessness of the electric part of the Weyl tensor, seen in Eq.(2.32), determines the following relations

$$K_{(4)} = \frac{\ell}{2} Tr \left(K^2_{(2)} \right) , \tag{2.55}$$

$$K_{(6)} = \frac{\ell}{2} Tr \left(K_{(2)} K_{(4)} \right) . \tag{2.56}$$

Solving the Einstein's equations in the FG frame, allows us to determine the asymptotic behavior of the extrinsic curvature. Notice that it is a radial expansion of intrinsic curvature invariants whose number of derivatives is analogous to the power of the radial coordinate. This property will be crucial for the derivation of the standard counterterm series starting from an expression that has an explicit dependence on the extrinsic curvature, namely, the Kounterterms.

2.2.2 Asymptotic fall-off of the Weyl tensor

The main objective of the first part of the thesis, is the comparison between two regularization schemes, which is translated at the level of their respective surface terms, the standard counterterms and the Kounterterms. The analysis in the most general case is cumbersome, but we can simplify the problem by imposing specific boundary conditions associated to the fall-off of the Weyl tensor. In what follows we define a specific class of spacetimes, called asymptotically conformally flat spaces, which correspond to the terrain of our analysis.

Starting from the second Bianchi indentity for the Weyl tensor, one gets

$$\nabla_{[\alpha} W^{\mu\nu}_{\beta\gamma]} \equiv \nabla_\alpha W^{\mu\nu}_{\beta\gamma} + \nabla_\beta W^{\mu\nu}_{\gamma\alpha} + \nabla_\gamma W^{\mu\nu}_{\alpha\beta} = 0 . \tag{2.57}$$

Using the GC relations [14], a series of differential equations for the independent components of the Weyl tensor is found, which cast the Bianchi identity in the form

$$
\begin{aligned}
\dot{W}^{ij}_{kl} &= D_l W^{ij}{}_k - D_k W^{ij}{}_l - K^i_m W^{mj}_{kl} + K^j_m W^{mi}_{kl} \\
&\quad + K^i_k E^j_l - K^j_k E^i_l - K^i_l E^j_k + K^j_l E^i_k \,, \\
\dot{W}_{ij}{}^k &= -D_i E^k_j + D_j E^k_i + K_{il} W^{kl}{}_j - K_{jl} W^{kl}{}_i - K^k_l W_{ij}{}^l \,, \\
0 &= D_{[m} W^{ij}_{kl]} - K^i_{[m} W_{kl]}{}^j + K^j_{[m} W_{kl]}{}^i \,, \\
0 &= D_{[l} W_{ij]}{}^k - K_{im} W^{km}_{jl} + K_{jm} W^{km}_{il} - K_{lm} W^{km}_{ij} \,,
\end{aligned} \tag{2.58}
$$

where D_m is the covariant derivative of the boundary metric h_{ij}.

It is evident that in order to determine the fall-off, the previous equations have to be solved asymptotically, assuming the asymptotic expansion of the extrinsic curvature (2.26). We are keeping only the first two terms, namely

$$
K^i_j\left(h\right) \simeq \frac{1}{\ell}\delta^i_j + \ell\rho\mathcal{S}^i_{(0)j} \simeq \frac{1}{\ell}\delta^i_j + \ell\mathcal{S}^i_j\left(h\right) \,. \tag{2.59}
$$

Moreover, considering the second order pole of the ALAdS spacetimes, the fall-off of the metric h_{ij} is of order $\mathcal{O}\left(\rho^{-1}\right)$ when expressed in FG coordinates or $h_{ij} = \mathcal{O}\left(e^{2r/\ell}\right)$ when written in Schwarzschild coordinates. For the curvatures, on the other hand, we get the following asymptotic behavior

$$
\begin{aligned}
\mathcal{R}^{ij}_{kl}\left(h\right) = \rho\mathcal{R}^{ij}_{kl}\left(g\right) = \mathcal{O}\left(e^{-2r/\ell}\right) \,, \\
\mathcal{S}^i_j\left(h\right) = \rho\mathcal{S}^i_j\left(g\right) = \mathcal{O}\left(e^{-2r/\ell}\right) \,.
\end{aligned}
$$

Taking into account only the leading order contribution from (2.59), the differential equations in (2.58) can be rewritten as

$$
\dot{W}^{ij}_{kl} \simeq D_l W^{ij}{}_k - D_k W^{ij}{}_l - \frac{2}{\ell} W^{ij}_{kl} + \frac{1}{\ell}\left(\delta^i_k E^j_l - \delta^j_k E^i_l - \delta^i_l E^j_k + \delta^j_l E^i_k\right) \,, \tag{2.60}
$$

$$
\dot{W}_{ij}{}^k \simeq -D_i E^k_j + D_j E^k_i + \frac{1}{\ell}\left(W^k{}_{ij} - W^k{}_{ji} - W_{ij}{}^k\right) \,, \tag{2.61}
$$

$$
D_{[m} W^{ij}_{kl]} \simeq \frac{1}{\ell}\left(\delta^i_{[m} W_{kl]}{}^j - \delta^j_{[m} W_{kl]}{}^i\right) \,, \tag{2.62}
$$

$$
D_{[i} W_{jl]}{}^k \simeq \frac{1}{\ell} W^k{}_{[ijl]} \,. \tag{2.63}
$$

Taking the trace of the Eq.(2.60) and substituting the EOM (2.31), we obtain the following relation that gives us the fall-off of the electric part of the Weyl tensor. More specifically, one gets that

$$
\dot{E}^i_k + \frac{d}{\ell} E^i_k \simeq -D_j W^{ij}{}_k \,. \tag{2.64}
$$

Restricting the asymptotic expansion of K_{ij} to the leading order, we get that

$$D_j W^{ij}{}_k = \mathcal{O}\left(e^{-\frac{nr}{\ell}}\right), \; n > d\,, \tag{2.65}$$

namely, the differential equation (2.64) is homogeneous. Hence the asymptotic behavior of E^i_j is

$$E^i_j \simeq E^i_{(0)j} e^{-dr/\ell} = \mathcal{O}\left(\rho^{d/2}\right)\,. \tag{2.66}$$

This condition defines the ACF spaces, that is of particular interest for the discussion below.

2.3 Counterterms and the EOM in ACF spaces

The asymptotic behavior of the bulk Weyl tensor in ACF spaces is given by the fall-off of its electric part, thus, $E^i_j = \mathcal{O}\left(e^{-dr/\ell}\right)$. As a consequence, the EOM we obtain in section 2.2.1 have to be modified. In this case, the Eq.(2.31) becomes

$$W^{ik}_{jk} = \mathcal{O}\left(e^{-dr/\ell}\right)\,, \tag{2.67}$$

or equivalently

$$\mathcal{R}^i_j\left(h\right) - KK^i_j + K^i_k K^k_j + \frac{d-1}{\ell^2}\delta^i_j = \mathcal{O}\left(e^{-dr/\ell}\right)\,. \tag{2.68}$$

In what follows we consider the covariant expansion of the extrinsic curvature as a function of the induced metric h_{ij} instead of $g_{(0)ij}$. The reason behind this choice is the fact that the covariant coefficients of the expansion of K, represent the eigenvectors of the dilatation operator.

It is known from [15], that the $(2p)$-derivative terms of the standard counterterm series are proportional to the relevant $(2p)$-derivative term of the radial canonical momentum expansion, given by

$$\mathcal{L}_{ct(2p)} = \frac{2}{d-2p}\pi_{(2p)}\,. \tag{2.69}$$

Based on the Hamiltonian formulation for the AdS bulk dynamics, the radial coordinate plays the role of time of the standard ADM formalism. In this context, the canonical momentum is the conjugate to the induced metric h_{ij} and acquires the form

$$\pi^{ij} = \frac{1}{2\kappa^2}\sqrt{-h}\left(Kh^{ij} - K^{ij}\right)\,. \tag{2.70}$$

Using the Hamilton-Jacobi formalism [16], it was proven that the counterterms series expansion are eigenfunctions of the dilatation operator

$$\delta_D = 2\int d^d x h_{ij} \frac{\delta}{\delta h_{ij}} \,. \tag{2.71}$$

On the same footing, the canonical momentum can be expanded asymptotically, where each term is acted by the dilatation operator as follows

$$\delta_D \pi^{ij}_{(2p)} = -2\left(p+1\right)\pi^{ij}_{(2p)} \,. \tag{2.72}$$

As a consequence, the extrinsic curvature admits the expansion

$$\begin{aligned} K^i_j = \sum_{p=0}^{\infty} K^i_{(2p)j} &= K^i_{(0)j}\left(h\right) + K^i_{(2)j}\left(h\right) + K^i_{(4)j}\left(h\right) + K^i_{(6)j}\left(h\right) + \dots \\ &= K^i_{(0)j} + \rho K^i_{(2)j}\left(g\right) + \rho^2 K^i_{(4)j}\left(g\right) + \rho^3 K^i_{(6)j}\left(g\right) + \dots \,, \end{aligned} \tag{2.73}$$

due to the action of δ_D.

Applying this into Eq.(2.68), we get a similar expression to the one derived before (section 2.2.1), considering that in ACF spaces the electric part of the Weyl tensor vanishes. On top of that, by inverting the series the terms $\mathcal{R}_{(1)}, \mathcal{R}_{(2)}$ of the Riemann and Ricci tensor expansion are absent, and the new formula for the EOM is written as

$$\begin{aligned} & \frac{\rho}{\ell}\left[\ell\mathcal{R}^i_j\left(g\right) - \left(d-2\right)K^i_{(2)j} - K_{(2)}\delta^i_j\right] + \\ + \; & \frac{\rho^2}{\ell}\left[\ell K^i_{(2)l}K^l_{(2)j} - \ell K^i_{(2)j}K_{(2)} - \left(d-2\right)K^i_{(4)j} - K_{(4)}\delta^i_j\right] \\ + \; & \frac{\rho^3}{\ell}\Big[-\left(d-2\right)K^i_{(6)j} - K_{(6)}\delta^i_j - \ell K_{(2)}K^i_{(4)j} - \ell K_{(4)}K^i_{(2)j} \\ + \; & 2\ell K^i_{(2)l}K^l_{(4)j}\Big] = 0 \,. \end{aligned} \tag{2.74}$$

Solving order by order, the various terms of the extrinsic curvature expansion become

$$K^i_{(0)j} = \frac{1}{\ell}\delta^i_j \,, \tag{2.75}$$

$$K^i_{(2)j} = \ell\mathcal{S}^i_j\left(g\right) \,, \tag{2.76}$$

$$\begin{aligned} K^i_{(4)j} &= \frac{\ell}{d-2}\left[K^i_{(2)k}K^k_{(2)j} - K_{(2)}K^i_{(2)j} + \frac{1}{2(d-1)}\delta^i_j\delta^{[i_1 i_2]}_{[j_1 j_2]}K^{j_1}_{(2)i_1}K^{j_2}_{(2)i_2}\right] \\ &= \frac{\ell^3}{d-2}\left[\mathcal{S}^i_k\mathcal{S}^k_j - \mathcal{S}\mathcal{S}^i_j - \frac{1}{2\left(d-1\right)}\left(\mathcal{S}^k_l\mathcal{S}^l_k - \mathcal{S}^2\right)\delta^i_j\right] \,, \end{aligned} \tag{2.77}$$

$$K^i_{(6)j} = \frac{\ell}{d-2}\Big[K^i_{(2)k}K^k_{(4)j} + K^i_{(4)k}K^k_{(2)j} - K_{(2)}K^i_{(4)j} - K_{(4)}K^i_{(2)j} + \\ + \frac{1}{d-1}\delta^i_j\delta^{[i_1i_2]}_{[j_1j_2]}K^{j_1}_{(2)i_1}K^{j_2}_{(4)i_2}\Big]\,. \tag{2.78}$$

Furthermore, taking the trace of (2.74), the following relations between traces are valid

$$K_{(4)} = -\frac{\ell}{2(d-1)}\delta^{[i_1i_2]}_{[j_1j_2]}K^{j_1}_{(2)i_1}K^{j_2}_{(2)i_2}\,, \tag{2.79}$$

$$K_{(6)} = -\frac{\ell}{(d-1)}\delta^{[i_1i_2]}_{[j_1j_2]}K^{j_1}_{(2)i_1}K^{j_2}_{(4)i_2}\,. \tag{2.80}$$

One can generalize this prescription and determine all the coefficients of the extrinsic curvature expansion to arbitrary order. Plugging the recursive relation (2.73) into the Eq.(2.68), one extends the Eq.(2.74), which now reads

$$\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\rho^p\delta^{[ik]}_{[mn]}K^m_{(2p-2l)k}K^n_{(2l)j} + \rho\left[(d-2)\,\mathcal{S}^i_j\,(g) + \mathcal{S}\,(g)\,\delta^i_j\right] + \\ + \frac{d-1}{\ell^2}\delta^i_j = \mathcal{O}\left(\rho^{d/2}\right)\,, \tag{2.81}$$

where the definition of the Schouten tensor of the boundary was taken into account. Evaluating for lower order p's we obtain the results shown in (2.75-2.78). Furthermore, for $1 \leq 2p \leq d$, the Eq.(2.81) can be written as

$$(d-2)\,K^i_{(2p)j} + K_{(2p)}\delta^i_j = \ell\sum_{l=1}^{p-1}\delta^{[ik]}_{[mn]}K^m_{(2l)k}K^n_{(2p-2l)j}\,. \tag{2.82}$$

Taking the trace we get

$$K_{(2p)} = -\frac{\ell}{2(d-1)}\sum_{l=1}^{p-1}\delta^{[ik]}_{[mn]}K^m_{(2l)i}K^n_{(2p-2l)k}\,, \tag{2.83}$$

or in the matrix representation

$$K_{(2p)} = -\frac{\ell}{2\,(d-1)}\sum_{l=1}^{p-1}\left\langle \mathbb{K}_{(2l)}\mathbb{K}_{(2p-2l)}\right\rangle\,. \tag{2.84}$$

For $p = 2, 3$ we recover the traces in Eqs. (2.79-2.80). Hence, the generic recursive formula that determines the coefficients of the extrinsic curvature expansion to arbitrary order, adopts the form

$$K^{i}_{(2p)j} = \frac{\ell}{d-2}\sum_{l=1}^{p-1}\left(\delta^{ik}_{mn}\, K^{m}_{(2l)k}K^{n}_{(2p-2l)j} + \frac{\delta^{i}_{j}}{2\,(d-1)}\,\left\langle \mathbb{K}_{(2l)}\mathbb{K}_{(2p-2l)}\right\rangle\right)$$
$$= \frac{\ell}{d-2}\sum_{l=1}^{p-1}\left(K^{i}_{(2l)k}K^{k}_{(2p-2l)j} - K^{n}_{(2p-2l)j}\,K_{(2l)} + \frac{\delta^{i}_{j}}{2\,(d-1)}\,\left\langle \mathbb{K}_{(2l)}\mathbb{K}_{(2p-2l)}\right\rangle\right). \tag{2.85}$$

From the Eqs.(2.69) and (2.70), we realize that the counterterm series is proportional to the trace of the extrinsic curvature of the relevant order. In this context, one of the main goals of the study is to certify the consistency of the Kounterterms with the aforementioned condition. The Eq.(2.85) is the key in this derivation, as it relates each term of the extrinsic curvature expansion and its trace, as a linear combination of the lower order terms.

Chapter 3

Kounterterms

3.1 Introduction

The EH action, when evaluated for a solution of the Einstein's equations, it is proportional to the volume of the corresponding spacetime. In case of a spacetime equipped with a negative cosmological constant, its volume tends to infinity, leading the on-shell action to diverge. The reason behind this behavior is the presence of the pole at the AdS boundary, expressed by the divergent conformal factor in the asymptotic sector in Eq.(2.13). Thus, as $\rho \to 0$, a series of IR divergences arise at the on-shell action.

Considering, that ALAdS spacetimes correspond to manifolds with a boundary, the EH action is augmented with a surface term that allows to obtain a well-defined variational principle considering Dirichlet boundary conditions for the metric h_{ij}. Hence, the action reads

$$I_{ren} = \frac{1}{16\pi G}\int_M d^{d+1}x\sqrt{\mathcal{G}}\left(R\left(\mathcal{G}\right) - 2\Lambda\right) - \frac{1}{8\pi G}\int_{\partial M} d^d x\sqrt{h}K\,, \tag{3.1}$$

where $K = K_i^i$ is the trace of the extrinsic curvature, corresponding to the foliation of the spacetime. The surface term is called the GH term. We can actually track-down the divergent pieces by evaluating the asymptotic solutions, given by the FG coefficients up to order $\rho^{d/2}$, previously derived at the section 2.2.1 . Furthermore, a radial cut-off scale is introduced in order to regulate the action, such that the bulk piece is evaluated for $\rho \geq \epsilon$, where $\epsilon > 0$, and the boundary term for $\rho = \epsilon$. Hence, the regulated action casts the form

$$S_{reg}\left(g_{(0)}\right) = \int d^d x\sqrt{-g_{(0)}}\left(\epsilon^{-d/2}\alpha_0 + \epsilon^{-d/2+1}\alpha_1 + \ldots + \epsilon^{-1}\alpha_{d-2} - \alpha_{d-2}\log\epsilon\right) + \mathcal{O}\left(\epsilon^0\right)\,.$$

Here, the α_n coefficients correspond to covariant expressions of the metric $g_{(0)}$. Notice the various order poles of the cut-off scale ϵ plus the logarithmic term that make the on-shell action divergent. Rendering the action finite corresponds to the cancellation of the divergences by adding proper surface terms, which are consistent with the diffeomorphism invariance of the theory. The general form of the renormalized action, reads

$$S_{ren}\left(g_{(0)}\right)=\lim_{\epsilon\to 0}\frac{1}{16\pi G}\left[S_{reg}\left(g_{(0)}\right)-\int d^dx\sqrt{-g_{(0)}}\left(\epsilon^{-d/2}\alpha_0+\epsilon^{-d/2+1}\alpha_1+\ldots\right.\right.$$
$$\left.\left.\ldots+\epsilon^{-1}\alpha_{d-2}-\alpha_{d-2}\log\epsilon\right)+\mathcal{O}\left(\epsilon^0\right)\right].$$

In [4], it was introduced a background-independent method for the elimination of the IR divergences. This regularizing scheme is called HR and is based on the addition of local covariant counterterms, namely surface terms which are functions of intrinsic quantities of the boundary, such as, the metric h_{ij}, the Riemann curvature $\mathcal{R}^{ij}_{kl}$ and its covariant derivatives. With this choice, we secure that the variational principle is well-defined considering Dirichlet boundary conditions for the metric h_{ij}.

In this prescription the general form of the renormalized AdS action reads

$$I_{ren}=\frac{1}{16\pi G}\int_M d^{d+1}x\sqrt{\mathcal{G}}\left(R\left(\mathcal{G}\right)+2\Lambda\right)-\frac{1}{8\pi G}\int_{\partial M}d^dx\sqrt{h}K+$$
$$+\int_{\partial M}\mathcal{L}_{ct}\left(h,\mathcal{R},D\mathcal{R}\right)\,, \tag{3.2}$$

where D is the covariant derivative with respect to the boundary connection. The leading order terms of the standard counterterm series can be written as

$$\mathcal{L}_{ct}\left(h,\mathcal{R},D\mathcal{R}\right)=\frac{\sqrt{-h}}{8\pi G}\left[\frac{d-1}{\ell}+\frac{\ell}{2\left(d-2\right)}\mathcal{R}+\right.$$
$$\left.+\frac{\ell^3}{2\left(d-2\right)^2\left(d-4\right)}\left(\mathcal{R}_{ij}\mathcal{R}^{ij}-\frac{d}{4\left(d-1\right)}\mathcal{R}^2\right)+\ldots\right]. \tag{3.3}$$

Even though it appears as a series of increasing in power Riemann curvature invariants of the boundary, $\mathcal{L}_{ct}$ corresponds to an expansion whose parameter is the radial coordinate ρ. This relation is manifest when considering that in the FG frame, the relation $\mathcal{R}^{ij}_{kl}\left(h\right)=\rho\mathcal{R}^{ij}_{kl}\left(g\right)$ holds. Thus, making higher power curvature invariants have faster fall-off than the lower power ones.

The counterterms $\mathcal{L}_{ct}$ are getting more complex going to higher order in $\mathcal{R}$ and to higher dimensions and their form is everything but compact. Moreover, the presence of a divergent conformal factor in h_{ij}, makes its variation not well defined $\left(\delta h_{ij}=\frac{1}{\rho}\delta g_{(0)ij}\right)$. What it really makes sense, in a mathematically rigorous way, is the fixing of either the conformal class $[h]$ or the source $g_{(0)ij}$.

On top of that, the $(2p)$-derivative term in the counterterms $\mathcal{L}_{(2p)}$ is proportional to the trace of the canonical momentum π_{ij}, as shown in [15]. For the radial foliation, the canonical momentum is a function of the extrinsic curvature

$$\pi^{ij} = \frac{1}{2\kappa^2}\sqrt{-h}\left(K^{ij} - Kh^{ij}\right) . \tag{3.4}$$

Thus, $\mathcal{L}_{(2p)}$ is actually proportional to the trace of the corresponding order of the extrinsic curvature expansion. This fact opens the possibility for alternative regularization schemes to HR, that have an explicit dependence on the extrinsic curvature of the boundary. In general, the presence of K^i_j is considered ill-defined from the point of view of the Dirichlet problem of h_{ij}, as variations of the first derivative of the metric arise. This is no longer true when different boundary conditions is considered.

A new regularizing scheme based on so called Kounterterms, was firstly introduced in [7]. It consists on the addition, on top of the Einstein-Hilbert action, of a surface term that is a given polynomial of both intrinsic and extrinsic curvatures. This new regularizing scheme provides a compact form for the cancellation of the divergences and its origin can be found on the Euler theorem itself. Despite the fact that the topological invariants can be found only on even-dimensional manifolds, the Kounterterms are generalized to odd dimensions as well [8].

Comparing the two regularizing schemes, we show that there is full agreement up to quartic order in the curvature for ACF manifolds. This equivalence allows us to introduce a closed recursive formula for the general counterterms series of order $2p$, common for both odd- and even-dimensional manifolds.

3.2 Kounterterms in ACF even-dimensional manifolds

From the previous discussion it is evident that the counterterms are proportional to the traces of the respective terms of the extrinsic curvature expansion $K^i_{(2p)j}$. This can be shown very easily, starting from Eq. (2.69) and substituting (2.70). Hence, the proportionality between the two expansion coefficients is obtained from the relation

$$\mathcal{L}_{\mathrm{ct}\,(2p)} = \frac{\sqrt{-h}}{8\pi G}\frac{d-1}{d-2p}\,\mathrm{Tr}\,\mathbb{K}_{(2p)} . \tag{3.5}$$

The same expression was derived in [16] considering a Hamiltonian approach to HR. It has to be noted that the expansion of the extrinsic curvature $K_{(2p)}$ represents different objects. On the one hand, in Eq. (3.5), it expresses the asymptotic expansion of K_{ij}, given in Eq.(2.26), while in [16] corresponds to the eigenvalues of the dilatation operator, previously seen in Eq.(2.73). The two objects are diferent, but there is 1-1 correspondence between them.

This feature combined with the fact that the variational problem for the induced metric h_{ij} is not well-defined, opens the discussion for alternative regularizing schemes.

In fact, the Kounterterms are an alternative counterterm series whose main characteristic is its explicit dependence on both intrinsic and extrinsic curvature of the boundary.

For a D-dimensional manifold M $(D = d + 1)$, the new formulation has the form

$$\tilde{I}_{ren} = I_{EH} + c_d \int\limits_{\partial M} d^d x \sqrt{-h} B_d \left(h, K, \mathcal{R}\right) \ , \tag{3.6}$$

where c_d is a coupling constant fixed by demanding either a well-defined variational principle, or the cancellation of the divergences of the Euclidean action for ALAdS spacetimes or the finiteness of the Noether charges. In any case, the specific value of c_d deals with these issues at once.

In the specific case of an even-dimensional manifold $(D = 2n)$, the surface term B_{2n-1} that is added has the form

$$\begin{aligned} B_{2n-1}\left(h, K, \mathcal{R}\right) = 2n\sqrt{h} \int_0^1 dt \delta^{[i_1 \ldots i_{2n-1}]}_{[j_1 \ldots j_{2n-1}]} K^{j_1}_{i_1} \left(\frac{1}{2} \mathcal{R}^{j_2 j_3}_{i_2 i_3}\left(h\right) - t^2 K^{j_2}_{i_2} K^{j_3}_{i_3} \right) \times \ldots \\ \ldots \times \left(\frac{1}{2} \mathcal{R}^{j_{2n-2} j_{2n-1}}_{i_{2n-2} i_{2n-1}}\left(h\right) - t^2 K^{j_{2n-2}}_{i_{2n-2}} K^{j_{2n-1}}_{i_{2n-1}} \right) \ , \end{aligned} \tag{3.7}$$

where the corresponding coupling constant is

$$c_{2n-1} = \frac{1}{16\pi G} \frac{\left(-1\right)^n \ell^{2n-2}}{n\left(2n-2\right)!} \ . \tag{3.8}$$

The value of the coupling constant is fixed demanding either a finite Euclidean action, or finite Noether charges or a well-defined variational principle. In any case, this specific value of c_d solves all the three problems at once.

The obvious advantage of the Kounterterms, in comparison to HR, is its compact format. Besides that, there is an even more fundamental reason, associated to the link between Kounterterms and topology. In order to make manifest the connection with topology, it is important to introduce the appropriate mathematical tools.

3.2.1 Topological origin of the Kounterterms

The Kounterterms are related to a gauge invariant extension of the CS forms, called transgression forms. The CS form $C_{2n+1}\left(A\right)$ is a differential form depending on a single connection A, whose exterior derivative defines a polynomial $P\left(F\right)$ in $2n+2$ dimensions, which is invariant under the action of the Lie group G on a vector space V.

In a more concrete way, considering the gauge connection $A = A_{\mu}^{I} T_{I} dx^{\mu}$ of the fibre bundle $G \times V$ with generators T_I, and its corresponding curvature two-form

$$F = dA + A \wedge A = \frac{1}{2} F_{\mu\nu}^{I} T_{I} dx^{\mu} dx^{\nu} \,, \tag{3.9}$$

the invariant polynomial $P(F)$ reads

$$P(F) \langle F^{n+1} \rangle \,. \tag{3.10}$$

The symbol $\langle \ldots \rangle$ expresses the totally symmetric invariant trace in the Lie algebra. At this point, the Chern-Weil theorem provides two interesting properties for the invariant polynomial. Firstly, it is a closed form, namely $dP(F) = 0$, and thus locally exact

$$P(F) = d\mathcal{C}_{2n+1}(A, F) \ , \tag{3.11}$$

where $\mathcal{C}_{2n+1}(A, F)$ is the CS density, that obtains the form

$$\mathcal{C}_{2n+1}(A, F) \equiv (n+1) \int_0^1 du \langle A F_u^n \rangle \,. \tag{3.12}$$

Here $A_u = uA$ and $F_u = dA_u + A_u \wedge A_u$, where u is an interpolating parameter. The CS densities are not truly invariant quantities under finite gauge transformations of the group G, as their variation results a non-vanishing surface term. A gauge-invariant extension of the CS forms is provided by the second property of the Chern-Weil theorem. According to this, the difference $P(\bar{F}) - P(F)$, where F and F' are curvatures corresponding to different gauge potentials A and $\bar{A}$ of the same homotopy class, is exact. Thus, the introduction of a second gauge potential is the key for the construction of a gauge-invariant quantity. In this case we get that

$$\langle F^{n+1} \rangle - \langle \bar{F}^{n+1} \rangle = d\mathcal{T}_{2n+1}(A, \bar{A}) \ , \tag{3.13}$$

where $\mathcal{T}_{2n+1}(A, \bar{A})$ is the transgression form, what is written as

$$\mathcal{T}_{2n+1}(A, \bar{A}) \equiv (n+1) \int_0^1 dt \langle (A - \bar{A}) F_t^n \rangle \,. \tag{3.14}$$

Here

$$A_t = tA + (1-t)\bar{A} \,, \tag{3.15}$$

is the interpolating gauge potential, and $F_t = dA_t + A_t \wedge A_t$ is the corresponding curvature. The transgression form is a gauge-invariant quantity but it comes with a price,namely, the presence of a second connection $\bar{A}$.

For the Lorentz Group in four dimensions $SO(3,1)$, the two gauge connections are $A = \frac{1}{2}\omega^{AB}J_{AB}$ and $A = \frac{1}{2}\bar{\omega}^{AB}J_{AB}$, while the invariant tensor constructed from the generators is $J_{AB}J_{CD} = \epsilon_{ABCD}$. Here A, B are tangent space indices with a range $0, 1, \ldots, D-1$, where D is the spacetime dimension. Hence, the invariant polynomial of the Lorentz group is the Gauss-Bonnet (GB) term $\mathcal{E}_4 = \epsilon_{ABCD}R^{AB}R^{CD}$, which is a topological invariant. The interpolating gauge connections obtains the form

$$A_t = \frac{1}{2}\omega_t^{AB}J_{AB} = \frac{1}{2}\left(\bar{\omega}^{AB} + t\theta^{AB}\right)J_{AB}\,, \tag{3.16}$$

where $\theta^{AB} = \omega^{AB} - \bar{\omega}^{AB}$ is the SFF. Hence, the curvature casts the form

$$F_t = \frac{1}{2}R_t^{AB}J_{AB} = \frac{1}{2}\left(\bar{R}^{AB} + t\bar{D}\theta^{AB} + t^2\theta^A_C\theta^{BC}\right)J_{AB}\,, \tag{3.17}$$

where $\bar{R}^{AB}$ and $\bar{D}$ are the curvature and the covariant derivative of the referential spin connection $\bar{\omega}^{AB} = \bar{\omega}_\mu^{AB}dx^\mu$, where $\mu = \rho, i$ are spacetime indices. As a result, the Eqs.(3.13,3.14) give

$$\mathcal{E}_4(R) - \bar{\mathcal{E}}_4(\bar{R}) = 2d\int_0^1 dt\epsilon_{ABCD}\theta^{AB}\left(\bar{R}^{CD} + t\bar{D}\theta^{CD} + t^2\theta^C_M\theta^{MD}\right)\,. \tag{3.18}$$

For the radial foliation of the form (2.23), the tangent space indices can be split as $A(1,a)$, where a denotes boundary indices and 1 expresses the radial direction. Thus, the vierbein can be written as

$$e^1 = Nd\rho\,, e^a = e^a_i dx^i\,. \tag{3.19}$$

For a Riemannian manifold, the spin connection is determined by the vierbein from the relation

$$\omega_\mu^{AB} = -e^{B\nu}\nabla_\mu e^A_\nu\,. \tag{3.20}$$

Hence, for Gauss-normal coordinates, one gets

$$\omega_\mu^{1a} = -K^a = K^a_i dx^i\,. \tag{3.21}$$

The spin connection of the boundary ω^{ab} has been omitted because it is not a Lorentz vector and lacks representation in the tensorial formalism. The problem is commonly solved [17] by introducing a second spin connection that corresponds to a fixed,locally product metric which is cobordant to the dynamical one. It casts the form

$$ds^2 = N^2(\rho)d\rho^2 + \bar{h}_{ij}(x)dx^i dx^j\,. \tag{3.22}$$

In this case, the connection coefficients $\bar{\omega}^{AB}$ read

$$\bar{\omega}^{1a} = 0\,, \bar{\omega}^{ab} = \omega^{ab}\,. \tag{3.23}$$

As a consequence, the SFF of the two cobordant metrics becomes

$$\theta^{1a} = -K^{a}\,, \theta^{ab} = 0\,. \tag{3.24}$$

Indeed, the introduction of the second spin connection, associated to a fixed product metric, leads to the construction of Lorentz covariant tensors. Actually, the presence of the SFF, instead of the spin connection itself, is compatible with Lorentz covariance.

Based on these considerations, it is straightforward to show that the Kounterterms B_3 in $D = 4$ dimensions, given by Eq. (3.7) for (n=2), can be written in a Lorentz covariant way as

$$B_3 = 2\epsilon_{ABCD}\theta^{AB}\left[R^{CD} + t^2\left(\theta^2\right)^{CD}\right]\,. \tag{3.25}$$

Moreover, taking into account the matching conditions between the dynamical and the product metric, the second term in the lhs and the second term in the RHS of the Eq.(3.18) vanish, while $\bar{R} = R$. As a consequence, one recovers the term B_3 what is now identified as the Transgression form of the Lorentz group $SO(3,1)$. Here, B_3 is alternatively called the second Chern form.

Furthermore, the significance of the second Chern form in a four-dimensional manifold is going even further. The Euler theorem for a manifold without a boundary states that

$$\int_{M_4} \mathcal{E}_4 = \int_{M_4} d^4x\sqrt{-\mathcal{G}}\left(R^{\alpha\beta}_{\mu\nu}R^{\mu\nu}_{\alpha\beta} - 4R^{\mu}_{\nu}R^{\nu}_{\mu} + R^2\right) = 32\pi^2\chi(M_4)\,, \tag{3.26}$$

where $\chi(M_4)$ is the Euler characteristic of the manifold, a number that is a topological invariant and doesn't change under continuous deformations of the space. This is the reason why the GB term $\mathcal{E}_4$, does not affect the dynamics of a four-dimensional spacetime. When a manifold with a boundary is considered, a boundary correction arises, which is identified as the the second Chern form B_3. More specifically, the Euler theorem now obtains the form

$$\int_{M_4} \mathcal{E}_4 = 32\pi^2\chi(M_4) + \int_{\partial M_4} B_3. \tag{3.27}$$

Thus, B_3 is equivalent to the GB up to the Euler characteristic. The Euler theorem can be generalized to an arbitrary, non-compact $D = 2n$-dimensional manifold, which states that

$$\int_{M_{2n}} d^{2n}x\mathcal{E}_{2n} = (4\pi)^n\, n!\chi\left(M\right) + \int_{\partial M_{2n}} d^{2n-1}xB_{2n-1}\,. \tag{3.28}$$

In complete analogy with the 4D case, B_{2n-1} is the transgression form of the Lorentz group $SO(2n-1,1)$, namely the n-th Chern form, and it is equivalent to the Euler term of the corresponding dimension.

This result makes manifest the deep connection between the Kounterterms and topological invariance. This formulation introduces the concept of topological regularization, and shows that the elimination of the divergences is not a trivial procedure but it is equivalent to the addition of a topological invariant of the Euler class on top of the EH-AdS action. As a consequence, one may rewrite the renormalized action (3.6) as

$$\tilde{I}_{ren} = \frac{1}{16\pi G}\int_M d^{2n}x\sqrt{g}[R - 2\Lambda + \alpha_{2n}\delta^{[\nu_1\ldots\nu_{2n}]}_{[\mu_1\ldots\mu_{2n}]} R^{\mu_1\mu_2}_{\nu_1\nu_2}\ldots R^{\mu_{2n-1}\mu_{2n}}_{\nu_{2n-1}\nu_{2n}}] \,, \tag{3.29}$$

where the coupling constant reads

$$\alpha_{2n} = (-1)^n \frac{\ell^{2n-2}}{2^n n\,(2n-2)!} \,. \tag{3.30}$$

The two formulas (3.29) and (3.6) are equivalent up to the Euler characteristic. Moreover, notice that the coupling constants α_{2n} and c_{2n-1} are equal up to a numerical factor $1/2^n$ due to the antisymmetric properties of the Riemann curvature and the use of the generalized delta. Hence, the coupling constant of the topological invariant is not arbitrary but obtains a fixed value, defined by the finiteness of the Euclidean action, the finiteness of the Noether charges or by demanding a well-defined variational principle.

3.2.2 Explicit calculation of the counterterms

Unlike the standard counterterm series $\mathcal{L}_{ct}(h)$, which depend on intrinsic quantities of the boundary (3.2), the n-th Chern form presents an explicit dependence on the extrinsic curvature of the boundary K_{ij}. Even if this difference seems difficult to circumvent, the asymptotic expansion of K_{ij} in Eq.(2.75-2.78), is the key that allows to compare the two regularizing schemes.

The first step towards this derivation, is by adding and subtracting the GH term in (3.6), in order to be able to compare with the standard holographic renormalization action I_{ren} given in Eq.(3.2). The renormalized action adopts the form

$$\tilde{I}_{ren} = I_{EH} - \frac{1}{8\pi G}\int_{\partial M} d^{2n-1}x\sqrt{-h}K + \int_{\partial M} d^{2n-1}x\tilde{\mathcal{L}}_{ct}(h,K,\mathcal{R}) \,, \tag{3.31}$$

where the counterterms are identified as

$$\tilde{\mathcal{L}}_{ct}(h,K,\mathcal{R}) = c_{2n-1}B_{2n-1} + \frac{1}{8\pi G}\sqrt{-h}K + \mathcal{O}(1) \,. \tag{3.32}$$

Moreover, the finite terms that arise in Eq.(3.78), are scheme dependent and they do not affect the holographic information on the boundary.

Our main purpose is to determine the coefficients of $\tilde{\mathcal{L}}_{ct}\left(h,K,\mathcal{R}\right)$ and compare it with the standard counterterm series $\mathcal{L}_{ct}\left(h,\mathcal{R},D\mathcal{R}\right)$. Substituting the exact expression of the n-th Chern form (3.7) in (3.32), the counterterms obtain the form

$$\begin{aligned}\tilde{\mathcal{L}}_{ct}\left(h,K,\mathcal{R}\right) = & \frac{\left(-1\right)^{n}\ell^{2n-2}\sqrt{-h}}{8\pi G\left(2n-2\right)!}\delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]}K^{j_1}_{i_1}\int_0^1 dt\left[\left(\frac{1}{2}\mathcal{R}^{j_2j_3}_{i_2i_3}-t^2K^{j_2}_{i_2}K^{j_3}_{i_3}\right)\times\ldots\right.\\ & \left.\ldots\times\left(\frac{1}{2}\mathcal{R}^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}-t^2K^{j_{2n-2}}_{i_{2n-2}}K^{j_{2n-1}}_{i_{2n-1}}\right)+\frac{\left(-1\right)^{n}}{\ell^{2n-2}}\delta^{j_2}_{i_2}\ldots\delta^{j_{2n-1}}_{i_{2n-1}}\right]. \qquad (3.33)\end{aligned}$$

One can reformulate this expression by introducing the bulk Weyl tensor from the equation (2.27), as

$$\begin{aligned}\tilde{\mathcal{L}}_{ct}\left(h,K,\mathcal{R}\right) = & \frac{\left(-1\right)^{n}\ell^{2n-2}\sqrt{-h}}{8\pi G\left(2n-2\right)!}\delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]}K^{j_1}_{i_1}\int_0^1 dt\left\{\left[\frac{1}{2}W^{j_2j_3}_{i_2i_3}+\left(1-t^2\right)K^{j_2}_{i_2}K^{j_3}_{i_3}-\frac{1}{\ell^2}\delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\right]\times\right.\\ & \ldots\times\left[\frac{1}{2}W^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}+\left(1-t^2\right)K^{j_{2n-2}}_{i_{2n-2}}K^{j_{2n-1}}_{i_{2n-1}}-\frac{1}{\ell^2}\delta^{j_{2n-2}}_{i_{2n-2}}\delta^{j_{2n-1}}_{i_{2n-1}}\right]+\\ & \left.+\frac{\left(-1\right)^{n}}{\ell^{2n-2}}\delta^{j_2}_{i_2}\ldots\delta^{j_{2n-1}}_{i_{2n-1}}\right\}. \qquad (3.34)\end{aligned}$$

In order to generate the standard counterterm series, we have to expand $\tilde{\mathcal{L}}_{ct}$ in powers of the radial coordinate ρ or in powers of the boundary intrinsic curvature. For this reason, we are, initially, restricting ourselves to the ACF case (2.66), a constraint that allows us to simplify the derivation by dropping the emerging Weyl tensor W^{ij}_{kl}.

After substituting the asymptotic expansion of the extrinsic curvature and considering the lowest-order approximation of the binomial expansion in the $(n-1)$ parentheses (Appendix C), the final form of the Kounterterm Lagrangian adopts the following form

$$\begin{aligned}\tilde{\mathcal{L}}_{ct}\left(h,K,\mathcal{R}\right) = & \frac{\ell^{2n-2}\sqrt{-h}}{8\pi G\left(2n-2\right)!}\delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]}\left(K^{j_1}_{(0)i_1}+\rho K^{j_1}_{(2)i_1}+\rho^2K^{j_1}_{(4)i_1}+\rho^3K^{j_1}_{(6)i_1}\right)\\ & \times\int_0^1 dt\left[\left(1-t^{2n-2}\right)M_0+\rho\Pi_2+\rho^2\Pi_4+\rho^3\Pi_6\right]. \qquad (3.35)\end{aligned}$$

Starting from (3.35) we identify the lowest order terms in the curvature and we compare them with the standard counterterms. A key feature in the comparison is the property (3.5), expressing the proportionality between the $(2p)$-derivative terms of the counterterms series and the extrinsic curvature coefficients.

The leading order contribution reads,

$$\begin{aligned}\tilde{\mathcal{L}}_{ct(0)} &= \frac{\ell^{2n-2}\sqrt{-h}}{8\pi G\left(2n-2\right)!}\delta^{[i_1\dots i_{2n-1}]}_{[j_1\dots j_{2n-1}]}\int\limits_0^1 dt\left(1-t^{2n-2}\right)K^{j_1}_{(0)i_1}M_0 \\ &= \frac{\sqrt{-h}}{8\pi G\ell\left(2n-2\right)!}\delta^{[i_1\dots i_{2n-1}]}_{[j_1\dots j_{2n-1}]}\int\limits_0^1 dt\left(1-t^{2n-2}\right)\delta^{j_1}_{i_1}\dots\delta^{j_{2n-1}}_{i_{2n-1}} \\ &= \frac{\sqrt{-h}}{8\pi G}\frac{2n-2}{\ell}=\mathcal{L}_{ct(0)}\,,\end{aligned} \tag{3.36}$$

and we see that the expressions (3.33) and (3.3) are really equivalent. Similarly, the next to the leading order term is written as

$$\begin{aligned}\tilde{\mathcal{L}}_{ct(2)} &= \rho\frac{\ell^{2n-2}\sqrt{-h}}{8\pi G\left(2n-2\right)!}\delta^{[i_1\dots i_{2n-1}]}_{[j_1\dots j_{2n-1}]}\int\limits_0^1 dt\left[\left(1-t^{2n-2}\right)K^{j_1}_{(2)i_1}M_0+K^{j_1}_{(0)i_1}\Pi_2\right] \\ &= \rho\frac{\ell^{2n-2}\sqrt{-h}}{8\pi G\left(2n-2\right)!}\delta^{[i_1\dots i_{2n-1}]}_{[j_1\dots j_{2n-1}]}\int\limits_0^1 dt\left[2\left(n-1\right)t^{2n-4}-\left(2n-1\right)t^{2n-2}+1\right]K^{j_1}_{(2)i_1}M_0 \\ &= \frac{\sqrt{-h}}{8\pi G}\frac{2n-2}{2n-3}K_{(2)}\left(h\right)=\mathcal{L}_{ct(2)}\,.\end{aligned} \tag{3.37}$$

Here, the equivalence seen in the leading order term, is extended to terms that they are linear in $K_{(2)}$, namely the Schouten tensor. The equivalence is manifest when taking into account the expansion seen in Eq.(3.5) for the standard counterterms. Writing the second derivative term as a function of the intrinsic curvature we get that

$$\tilde{\mathcal{L}}_{ct(2)}=\frac{\ell\sqrt{-h}}{8\pi G}\frac{1}{2(2n-3)}\mathcal{R}\left(h\right)\,. \tag{3.38}$$

Summing the contributions which are quadratic order in the curvature, one gets

$$
\begin{aligned}
\tilde{\mathcal{L}}_{ct(4)} &= \rho^2 \frac{\ell^{2n-2}\sqrt{-h}}{8\pi G\,(2n-2)!}\delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]}\int\limits_0^1 dt\left[K^{j_1}_{(0)i_1}\Pi_4 + K^{j_1}_{(2)i_1}\Pi_2 + \left(1-t^{2n-2}\right)K^{j_1}_{(4)i_1}M_0\right] \\
&= \frac{\sqrt{-h}}{8\pi G}\frac{1}{2n-3}\left[(2n-2)\,K_{(4)}\,(h) - \frac{\ell}{2n-5}\delta^{[i_1 i_2]}_{[j_1 j_2]}K^{j_1}_{(2)i_1}\,(h)\,K^{j_2}_{(2)i_2}\,(h)\right]. \qquad (3.39)
\end{aligned}
$$

In the section 2.2.3 we showed that the EOM provide relations between traces of the various terms of the extrinsic curvature expansion. Thus, taking into account the Eq.(2.79), the four-derivative counterterm is proportional to the trace of $K_{(4)ij}$ and adopts the form

$$
\tilde{\mathcal{L}}_{ct(4)} = \frac{\sqrt{-h}}{8\pi G}\frac{2n-2}{2n-5}K_{(4)} = \mathcal{L}_{ct(4)}\,, \qquad (3.40)
$$

which is the correct result. This formula when reexpressed in terms of the boundary curvature from Eq.(2.77), reads

$$
\tilde{\mathcal{L}}_{ct(4)} = \frac{\sqrt{-h}}{8\pi G}\frac{\ell^3}{2\,(2n-3)^2\,(2n-5)}\left(\mathcal{R}^i_j\mathcal{R}^j_i - \frac{2n-1}{4\,(2n-2)}\mathcal{R}^2\right). \qquad (3.41)
$$

Going to higher orders, the terms that contribute are cubic in curvature, as it is shown below

$$
\begin{aligned}
\tilde{\mathcal{L}}_{ct(6)} = \rho^3 \frac{\ell^{2n-2}\sqrt{-h}}{8\pi G\,(2n-2)!}\delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]}\int_0^1 dt\,\Big[K^{j_1}_{(0)i_1}\Pi_6 + K^{j_1}_{(2)i_1}\Pi_4 + \\
+ K^{j_1}_{(4)i_1}\Pi_2 + \left(1-t^{2n-2}\right)K^{j_1}_{(6)i_1}M_0\Big]\,. \qquad (3.42)
\end{aligned}
$$

Evaluating the parametric integrals we obtain the following symmetric traces in the expression

$$
\begin{aligned}
\tilde{\mathcal{L}}_{ct(6)} = \frac{\sqrt{-h}}{8\pi G}\frac{1}{2n-3}\Big[(2n-2)\,K_{(6)} - \frac{2}{2n-5}\ell\delta^{[i_1 i_2]}_{[j_1 j_2]}K^{j_1}_{(2)i_1}K^{j_2}_{(4)i_2} + \\
+ \frac{\ell^2}{(2n-5)\,(2n-7)}\delta^{[i_1 i_2 i_3]}_{[j_1 j_2 j_3]}K^{j_1}_{(2)i_1}K^{j_2}_{(2)i_2}K^{j_3}_{(2)i_3}\Big]\,. \qquad (3.43)
\end{aligned}
$$

We already know from Eq.(2.80), that the second term in $\tilde{\mathcal{L}}_{ct(6)}$ is proportional to the trace of $K_{(6)}$. In order to see if there is an analogous relation for the third term, we multiply $K^i_{(2)j}$ with Eq.(2.77), which can be written as

$$K^j_{(2)i}K^i_{(4)j} = \frac{\ell}{d-2}\left[K^j_{(2)i}K^i_{(2)k}K^k_{(2)j} - K^j_{(2)i}K_{(2)}K^i_{(2)j} + \right.$$
$$\left. + \frac{1}{2(d-1)}K_{(2)}\delta^{[i_1 i_2]}_{[j_1 j_2]}K^{j_1}_{(2)i_1}K^{j_2}_{(2)i_2}\right] . \tag{3.44}$$

The trace of $K_{(6)}$ can equivalently be expressed as

$$\begin{aligned} K_{(6)} &= -\frac{\ell}{d-1}\delta^{[i_1 i_2]}_{[j_1 j_2]}K^{j_1}_{(2)i_1}K^{j_2}_{(4)i_2} = -\frac{\ell}{d-1}\left(K_{(2)}K_{(4)} - K^i_{(2)j}K^j_{(4)i}\right) \\ &= \frac{\ell^2}{2(d-1)\,(d-2)}(K^3_{(2)} + 2K^j_{(2)i}K^i_{(2)k}K^k_{(2)j} - 3K_{(2)}K^j_{(2)i}K^i_{(2)j}) \\ &= \frac{\ell^2}{2(d-1)\,(d-2)}\delta^{[i_1 i_2 i_3]}_{[j_1 j_2 j_3]}K^{j_1}_{(2)i_1}K^{j_2}_{(2)i_2}K^{j_3}_{(2)i_3} , \end{aligned} \tag{3.45}$$

where Eqs.(3.44) and (2.79) used passing from the first to the second line. Hence, all the terms of Eq.(3.43) are proportional to the trace of $K_{(6)}$, and the final values of the cubic in curvature counterterms reads

$$\tilde{\mathcal{L}}_{ct(6)} = \frac{\sqrt{-h}}{8\pi G}\frac{2n-2}{2n-7}K_{(6)} = \mathcal{L}_{ct(6)} . \tag{3.46}$$

Note that the proportionality between the Kounterterms $\tilde{\mathcal{L}}_{ct}$ and the coefficients of the extrinsic curvature expansion holds at least up to cubic order in the curvature. An extension of this feature to arbitrary order will allow us determine the counterterms to all orders due to the compact form of the Chern form.

The cubic order counterterms can be rewritten as a function of intrinsic quantities of the boundary, as follows

$$\tilde{\mathcal{L}}_{ct(6)} = \frac{\sqrt{-h}}{8\pi G}\frac{\ell^5}{(2n-3)^4\,(2n-7)}\left[\mathcal{R}^k_j\mathcal{R}^j_i\mathcal{R}^i_k - \frac{3\,(2n-1)}{8\,(n-1)}\mathcal{R}\mathcal{R}^i_j\mathcal{R}^j_i + \right.$$
$$\left. + \frac{4n^2+4n-7}{64\,(n-1)^2}\mathcal{R}^3\right] . \tag{3.47}$$

From this highly non-trivial derivation, we conclude that at least up to the sixth-derivative terms the two regularizing schemes are equivalent, that is,

$$\tilde{I}_{ren} = I_{ren} + \mathcal{O}\left(R^4\right) . \tag{3.48}$$

3.2.3 Towards a general formula for the counterterms

Going to higher-order in the standard HR is a really complicated procedure. As we have already seen, going to higher dimensions, where the higher order counterterms are becoming relevant, increases the complexity of the expressions even in the simplified ACF case.

The equivalence with the Kounterterms for terms up to cubic order in the curvature, opens the possibility of extending the matching beyond the six-derivative terms and obtain a systematic formula for the derivation of the counterterms. Indeed, below we propose a general formulation for the counterterms of order $2p$, when $p > 2$.

The first step towards this derivation is through the matrix representation of the counterterms. In this case, the Eq. (3.34) adopts the form

$$\tilde{\mathcal{L}}_{\rm ct} = -\frac{\sqrt{-h}}{8\pi G}\left[\frac{\left(-\ell^2\right)^{n-1}}{(2n-2)!}\left\langle \mathbb{K}\int_0^1 dt\left(\frac{1}{2}\mathbb{W}+\left(1-t^2\right)\mathbb{K}^2-\frac{1}{\ell^2}\mathbb{I}^2\right)^{n-1}\right\rangle - \langle\mathbb{K}\rangle\right]. \tag{3.49}$$

For ACF spaces, the Weyl tensor is dropped and the general formula now reads

$$\tilde{\mathcal{L}}_{\rm ct} = -\frac{\sqrt{-h}}{8\pi G}\left[\frac{1}{(2n-2)!}\int_0^1 dt\left\langle \mathbb{K}\left(\mathbb{I}^2-\ell^2\left(1-t^2\right)\mathbb{K}^2\right)^{n-1}\right\rangle - \langle\mathbb{K}\rangle\right]. \tag{3.50}$$

The polynomial expansion that appears, can now be written in its most general form as

$$\begin{aligned}&\left\langle \mathbb{K}\left(\mathbb{I}^2-\ell^2\left(1-t^2\right)\mathbb{K}^2\right)^{n-1}\right\rangle \\ =\;& (n-1)!\sum_{k=0}^{n-1}\frac{\left(-\ell^2\right)^k(2n-2k-2)!}{(n-1-k)!k!}\left(1-t^2\right)^k\left\langle\mathbb{K}^{2k+1}\right\rangle.\end{aligned} \tag{3.51}$$

The first terms of this expansion is shown in the Eq.(C.3). Here, the symbol $\langle\ldots\rangle$ indicates the presence of symmetric traces of the extrinsic curvature. As completely symmetric traces are defined products of matrices whose indices are saturated by the generalized Cronecker delta. Thus, if $\mathbb{A} = \left[A^i_j\right]$ is a $d\times d$ matrix, the symmetric trace of the tensorial product of p matrices $\mathbb{A}$ is given by

$$\delta^{[j_1\ldots j_p]}_{[i_1\ldots i_p]}A^{i_1}_{j_1}\ldots A^{i_p}_{j_p} = \langle\mathbb{A}^p\rangle\,. \tag{3.52}$$

A really interesting property of Eq.(3.51) is that the $k = 0$ term of the polynomial expansion cancels completely the contribution coming from the GH term, namely, the linear term in the extrinsic curvature given in $\tilde{\mathcal{L}}_{ct}$. Thus, performing the parametric integration, the series reads

$$\tilde{\mathcal{L}}_{\rm ct} = -\frac{\sqrt{-h}}{8\pi G}\frac{(n-1)!}{(2n-2)!}\sum_{k=1}^{n-1}\frac{\left(-4\ell^2\right)^k k!\,(2n-2k-2)!}{(n-1-k)!\,(2k+1)!}\left\langle \mathbb{K}^{2k+1}\right\rangle . \tag{3.53}$$

Once more, the key step is the asymptotic expansion of the extrinsic curvature. Unlike the last subsection, where we performed the expansion to the lowest order, here we consider the general binomial expansion of $\mathbb{K}^{2k+1}$, which can be written as

$$\begin{aligned}\left\langle \mathbb{K}^{2k+1}\right\rangle = \left(\mathbb{K}_{(0)}+\mathbb{K}_*\right)^{2k+1} &= \sum_{m_0=0}^{2k+1}\binom{2k+1}{m_0}\left\langle \mathbb{K}_{(0)}^{2k+1-m_0}\mathbb{K}_*^{m_0}\right\rangle \\ = \sum_{m_0=0}^{2k+1}\sum_{m_1=0}^{m_0}\sum_{m_2=0}^{m_0-m_1}\cdots\sum_{m_p=0}^{m_0-(m_1+\cdots+m_{p-1})}&\binom{2k+1}{m_0}\binom{m_0}{m_1}\binom{m_0-m_1}{m_2}\binom{m_0-m_1-m_2}{m_3}\cdots \\ \cdots\binom{m_0-m_1-m_2-\cdots-m_{p-1}}{m_p}&\left\langle \mathbb{K}_{(0)}^{2k+1-m_0}\mathbb{K}_{(2)}^{m_1}\mathbb{K}_{(4)}^{m_2}\cdots\mathbb{K}_{(2p)}^{m_p}\right\rangle . \end{aligned} \tag{3.54}$$

The integers appearing in the expansion are subjected to the following constraints,

$$\begin{aligned} m_0 &= m_1+m_2+\ldots+m_p\,, \\ p &= m_1+2m_2+\ldots+pm_p\,, \end{aligned} \tag{3.55}$$

while the binomial coefficients can be further simplified as

$$\begin{aligned}&\binom{2k+1}{m_0}\binom{m_0}{m_1}\binom{m_0-m_1}{m_2}\binom{m_0-m_1-m_2}{m_3}\cdots \\ &\cdots\binom{m_0-m_1-m_2-\cdots-m_{p-1}}{m_p} = \frac{(2k+1)!}{(2k+1-m_0)!m_1!\cdots m_p!} . \end{aligned} \tag{3.56}$$

Summing all these contributions, the symmetric trace product of the extrinsic curvature is expressed as a function of its asymptotic expansion coefficients, and acquires the form

$$\left\langle \mathbb{K}^{2k+1}\right\rangle_{(2p)} = \sum_{\{m_i\}}\frac{(2k+1)!}{(2k+1-m_0)!m_1!\cdots m_p!}\left\langle \mathbb{K}_{(0)}^{2k+1-m_0}\mathbb{K}_{(2)}^{m_1}\mathbb{K}_{(4)}^{m_2}\cdots\mathbb{K}_{(2p)}^{m_p}\right\rangle . \tag{3.57}$$

Substituting this result into (3.53), we get that

$$\tilde{\mathcal{L}}_{\mathrm{ct}\,(2p)} = \frac{\sqrt{-h}}{8\pi G} \sum_{\{m_i\}} \frac{\ell^{m_0-1} C^{(2n-1)}_{m_1,\ldots,m_p}}{m_1! \cdots m_p!} \left\langle \mathbb{K}^{m_1}_{(2)} \mathbb{K}^{m_2}_{(4)} \cdots \mathbb{K}^{m_p}_{(2p)} \right\rangle , \tag{3.58}$$

where the coefficients read

$$C^{(2n-1)}_{m_1,\ldots,m_p} = \sum_{k=1}^{n-1} \frac{(-4)^k (n-1)!\,(2n-1-m_0)!k!}{(2n-2)!\,(n-1-k)!\,(2k)!\,(2k+1-m_0)!} . \tag{3.59}$$

Hence, exploiting the properties of the Kounterterms we have shown that there is a generic and closed expression for the $(2p)$-order terms of the counterterm series.

For the two schemes to match, the following relation hold to arbitrary order

$$\left\langle \mathbb{K}^{m_1}_{(2)} \mathbb{K}^{m_2}_{(4)} \cdots \mathbb{K}^{m_p}_{(2p)} \right\rangle \propto \left\langle \mathbb{K}_{(2p)} \right\rangle . \tag{3.60}$$

For $0 \leq p \leq 3$, we recover the counterterms $\tilde{\mathcal{L}}_{ct}$ up to sixth order, previously derived in section 3.3.1, where it has been shown that the matching is valid.

In the case of $p = 4$, which defines the 8th-derivative counterterm, the Eq. (3.58) is written as

$$\begin{aligned} \tilde{\mathcal{L}}_{\mathrm{ct}(8)} &= \frac{\sqrt{-h}}{8\pi G} \frac{1}{d-2} \Bigg[(d-1) \left\langle \mathbb{K}_{(8)} \right\rangle - \frac{2\ell}{d-4} \left\langle \mathbb{K}_{(2)} \mathbb{K}_{(6)} \right\rangle - \frac{\ell}{d-4} \left\langle \mathbb{K}^2_{(4)} \right\rangle \\ &+ \frac{3\ell^2}{(d-4)(d-6)} \left\langle \mathbb{K}^2_{(2)} \mathbb{K}_{(4)} \right\rangle - \frac{\ell^3}{(d-4)(d-6)(d-8)} \left\langle \mathbb{K}^4_{(2)} \right\rangle \Bigg] , \end{aligned} \tag{3.61}$$

where $d = 2n - 1$. Considering the recursive relation in Eq.(2.83), the trace of $K_{(8)}$ reads

$$\left\langle \mathbb{K}_{(8)} \right\rangle = -\frac{\ell}{2(d-1)} \left(2 \left\langle \mathbb{K}_{(2)} \mathbb{K}_{(6)} \right\rangle + \left\langle \mathbb{K}^2_{(4)} \right\rangle \right) , \tag{3.62}$$

leading to the simplification of the above relation

$$\begin{aligned} \tilde{\mathcal{L}}_{\mathrm{ct}(8)} &= \frac{\sqrt{-h}}{8\pi G} \frac{1}{d-4} \Bigg[(d-1) \left\langle \mathbb{K}_{(8)} \right\rangle + \\ &+ \frac{\ell^2}{(d-2)(d-6)} \left(3 \left\langle \mathbb{K}^2_{(2)} \mathbb{K}_{(4)} \right\rangle - \frac{\ell}{d-8} \left\langle \mathbb{K}^4_{(2)} \right\rangle \right) \Bigg] . \end{aligned} \tag{3.63}$$

Multiplying the Eq.(2.78) by $K^i_{(2)j}$ and the Eq.(2.77) by $K^i_{(2)j}$ and $K^i_{(4)j}$, we derive the traces

$$\begin{aligned}
\mathrm{Tr}\left(\mathbb{K}_{(2)}\mathbb{K}_{(6)}\right) &= \frac{\ell}{d-2}\left\langle \mathbb{K}_{(2)}^{2}\mathbb{K}_{(4)}\right\rangle + \left\langle \mathbb{K}_{(2)}\right\rangle\left\langle \mathbb{K}_{(6)}\right\rangle ,\\
\mathrm{Tr}\left(\mathbb{K}_{(2)}^{2}\mathbb{K}_{(4)}\right) &= -\frac{\ell}{6\,(d-2)}\left\langle \mathbb{K}_{(2)}^{4}\right\rangle + \frac{2\,(d-1)}{\ell}\left\langle \mathbb{K}_{(4)}\right\rangle^{2} + \left\langle \mathbb{K}_{(2)}\right\rangle^{2}\left\langle \mathbb{K}_{(4)}\right\rangle \\
&\quad + \frac{d-1}{3\ell}\left\langle \mathbb{K}_{(2)}\right\rangle\left\langle \mathbb{K}_{(6)}\right\rangle ,\\
\mathrm{Tr}\left(\mathbb{K}_{(4)}^{2}\right) &= \frac{\ell}{2\,(d-2)}\left\langle \mathbb{K}_{(2)}^{2}\mathbb{K}_{(4)}\right\rangle + \left\langle \mathbb{K}_{(4)}\right\rangle^{2} .
\end{aligned} \tag{3.64}$$

Combining (3.62) with (3.64), we arrive to

$$\left\langle \mathbb{K}_{(2)}^{2}\mathbb{K}_{(4)}\right\rangle = \frac{4\,(d-1)\,(d-2)}{5\ell^{2}}\left\langle \mathbb{K}_{(8)}\right\rangle , \tag{3.65}$$

while the rest of the traces derived allow us to calculate

$$\left\langle \mathbb{K}_{(2)}^{4}\right\rangle = 12\,(d-1)\,(d-2)\left[\left\langle \mathbb{K}_{(2)}\right\rangle^{4} - \frac{1}{3\ell^{2}}\left\langle \mathbb{K}_{(2)}\right\rangle\left\langle \mathbb{K}_{(6)}\right\rangle - \frac{d-2}{\ell^{3}}\left\langle \mathbb{K}_{(8)}\right\rangle\right] . \tag{3.66}$$

Hence, the 8th derivative term on the counterterm series acquires the following form

$$\begin{aligned}
\tilde{\mathcal{L}}_{\mathrm{ct}(8)} &= \frac{\sqrt{-h}}{8\pi G}\,\frac{d-1}{d-4}\left[\frac{5d^{2}+2d+24}{5\,(d-6)\,(d-8)}\left\langle \mathbb{K}_{(8)}\right\rangle + \right.\\
&\quad \left. + \frac{12\ell^{3}}{(d-6)\,(d-8)}\left(\frac{1}{3\ell^{2}}\left\langle \mathbb{K}_{(2)}\right\rangle\left\langle \mathbb{K}_{(6)}\right\rangle - \left\langle \mathbb{K}_{(2)}\right\rangle^{4}\right)\right] .
\end{aligned} \tag{3.67}$$

The second line turns out not to be proportional to $\left\langle \mathbb{K}_{(8)}\right\rangle$. This fact indicates a mismatch between the standard holographic renormalization and the Kounterterms in ACF spaces of ten dimensions and higher.

3.3 Kounterterms in ACF odd-dimensional manifolds

Generalizing the previous prescription to odd-dimensional bulk spacetimes is not a straightforward process. As we saw in the section 3.2, the Kounterterms are actually obtained using topological invariants of the Euler class, which in odd dimensions do not exist. Nevertheless, in section 3.2.1 it is shown that the surface term (3.7) is identified as the transgression form of the Lorentz group $SO(2n-1,1)$, which is not restricted by the parity of the dimensionality of the manifold.

Based on this feature a regularizing technique, similar to the one proposed in even-dimensions, was formulated for odd-dimensional manifolds. It is used for first time in CS-AdS gravity and is generalized later to Einstein-AdS gravity. It consists on the addition of the surface term

$$B_{2n} = 2n\sqrt{-h}\int_0^1 dt \int_0^t ds \delta^{[i_1\cdots i_{2n}]}_{[j_1\cdots j_{2n}]} K^{j_1}_{i_1}\delta^{j_2}_{i_2}\left(\frac{1}{2}\mathcal{R}^{j_3j_4}_{i_3i_4} - t^2 K^{j_3}_{i_3}K^{j_4}_{i_4} + \frac{s^2}{\ell^2}\delta^{j_3}_{i_3}\delta^{j_4}_{i_4}\right)\times \dots$$
$$\dots \times \left(\frac{1}{2}\mathcal{R}^{j_{2n-1}j_{2n}}_{i_{2n-1}i_{2n}} - t^2 K^{j_{2n-1}}_{i_{2n-1}}K^{j_{2n}}_{i_{2n}} + \frac{s^2}{\ell^2}\delta^{j_{2n-1}}_{i_{2n-1}}\delta^{j_{2n}}_{i_{2n}}\right), \tag{3.68}$$

multiplied by a coupling constant

$$c_{2n} = \frac{1}{16\pi G}\frac{(-1)^n \ell^{2n-2}}{2^{2n-2} n (n-1)!^2}. \tag{3.69}$$

The value of c_{2n} is fixed, leading to a finite value for the Euclidean action and the Noether charges, and a well-defined variational principle. This boundary term cancels the IR divergences and regularizes the EH action. The formulation is consistent with the generic Kounterterms prescription given in Eq.(3.6), due to the explicit dependence of B_{2n} from the extrinsic curvature.

The main difference with respect to even bulk dimensions is the presence of a double parametric integration in Eq.(3.68). This fact indicates the different geometric structure of the boundary, which is evident from the presence of the logarithmic term in the FG expansion, in Eq.(2.14), for even-dimensional boundaries.

Furthermore, the geometrical character of the cancellation of divergences, that becomes manifest when applying the Kounterterms scheme in even-dimensional manifolds, is still valid. The difference can be spotted in the presence of CS densities instead of topological invariants. The connection of the boundary term B_{2n} with geometry is given with more details in the next section.

3.3.1 Geometrical origin of the Kounterterms in odd-dimensions

The key in the geometrical interpretation of the Kounterterms is the nature of the transgression forms. As we already saw in section 3.2.1, the introduction of a second gauge connection was the price we had to pay in order to construct a gauge-invariant extension of the CS densities,i.e., the transgression forms. The presence of the parametric integration and the restriction of the gauge connections to be parts of the same homotopy class, indicate the action of the Cartan homotopy operator.

Starting from the definition of the interpolating gauge potentials given in Eq.(3.15) and the respective curvature $F_t = dA_t + A_t \wedge A_t$, the action of the Cartan homotopy operator k_{01} in a polynomial $S(F_t, A_t)$ is given by

$$k_{01} S\left(F_t, A_t\right) = \int_0^1 dt \ell_t S\left(F_t, A_t\right) \,, \tag{3.70}$$

where

$$\ell_t A_t = 0 \,, \ell_t F_t = A - \bar{A} \,, \tag{3.71}$$

indicate the action of the operator ℓ_t. Moreover, integrating the following relation from 0 to 1

$$\left(d\ell_t + \ell_t d\right) S\left(F_t, A_t\right) = \frac{\partial}{\partial t} S\left(F_t, A_t\right) \,, \tag{3.72}$$

one gets the Cartan's homotopy formula, which acquires the form

$$\left(dk_{01} + k_{01} d\right) S\left(F_t, A_t\right) = S\left(F, A\right) - S\left(\bar{F}, \bar{A}\right) \,. \tag{3.73}$$

It is straightforward to show that for $S\left(F, A\right) = P(F)$, the second term in the LHS of Eq.(3.73) vanishes, because the invariant polynomial is a closed form, thus recovering the Eq.(3.13). As a result, the first term in the LHS corresponds to the transgression form that, when evaluated for the Lorentz group, one recovers the Kounterterms for even-dimensional manifolds B_{2n-1}. Hence, $k_{01} P(F) = B_{2n-1}$ when $A_\mu = \omega_\mu^{AB}$.

The question that arises at this point is the possibility to generalize the aforementioned scheme, considering that the Kounterterms are interpreted as the action of the Cartan homotopy operator on the Lorentz invariant polynomial, to polynomials of different type that would allow to extend the Kounterterms to odd dimensions.

Indeed, in case $S\left(F, A\right) = \mathcal{C}_{2n+1}$, the second term in the LHS of Eq.(3.73) gives the action of the homotopy operator on the corresponding invariant polynomial, due to Eq.(3.11). From the previous discussion, the action of k_{01} on the invariant polynomial is identified as the transgression form. Thus, the Eq. (3.73) when evaluated for the CS density gives

$$\mathcal{T}_{2n+1} = \mathcal{C}_{2n+1}\left(A, F\right) - \bar{\mathcal{C}}_{2n+1}\left(\bar{A}, \bar{F}\right) + d\Xi_{2n}\left(A, F, \bar{A}, \bar{F}\right) \,. \tag{3.74}$$

One identifies the surface term Ξ_{2n} as the action of the homotopy operator on the CS density, which reads

$$\Xi_{2n}\left(A, F, \bar{A}, \bar{F}\right) \equiv k_{01} \mathcal{C}_{2n+1} = n\left(n+1\right) \int_0^1 ds \int_0^1 dts \langle A_t \left(A - \bar{A}\right) F_{st}^{n-1} \rangle \,, \tag{3.75}$$

where $F_{st} = sF_t + s\left(s-1\right) A_t^2$.

The connection of the AdS group $SO(d-1,2)$ obtains the form

$$A = \frac{1}{2} \omega^{AB} J_{AB} + \frac{e^a}{\ell} P_A \,, \tag{3.76}$$

where J_{ab} correspond to Lorentz rotations and P_a to AdS translations. The corresponding curvature reads

$$F = \frac{1}{2}\left(R^{ab} + \frac{e^a e^b}{\ell^2}\right) J_{ab} + T^a P_a \,, \tag{3.77}$$

where $T^a = De^a$ is the torsion two-form. For Riemannian manifolds, the torsion vanishes.

In [18] it was shown that Ξ_{2n} matches B_{2n} in Eq.(3.68) when evaluated for the AdS group and written in tensorial form. In order to do so, the introduction of a cobordant, locally product metric with matching conditions of the form (3.24) for the SFF, is required. In this case, the surface term cancels the IR divergences arising in CS-AdS gravity.

Surprisingly enough, the same boundary term eliminates the divergences appearing at the asymptotic sector of EH-AdS gravity, as it is going to be shown explicitly in the next section. What is really interesting here, is that the cancellation of the divergences is coming from terms that are directly related to the action of the Cartan homotopy operator. That is, the reason why the Kounterterms have a profound geometrical character.

3.3.2 Explicit calculation of the counterterms

A derivation of the counterterms is no different from the one used in the even-dimensional case. More specifically, starting from the EH-AdS action for a $(2n+1)$-dimensional manifold, we add and subtract the GH term. Hence, the counterterm series coming from the Kounterterms B_{2n}, is written as

$$\tilde{\mathcal{L}}_{ct}\left(h, K, \mathcal{R}\right) = c_{2n} B_{2n} + \frac{1}{8\pi G}\sqrt{-h}K + \mathcal{O}\left(1\right) \,. \tag{3.78}$$

After plugging in the formula (3.68) in Eq.(3.78) and replacing the Riemann tensor of the boundary by the bulk Weyl tensor, the above relation becomes

$$\tilde{\mathcal{L}}_{ct}\left(h, K, \mathcal{R}\right) = \frac{\sqrt{-h}}{8\pi G}\frac{(-1)^n \ell^{2n-2}}{2^{2n-2}(n-1)!^2}\delta^{[i_1 \cdots i_{2n-1}]}_{[j_1 \cdots j_{2n-1}]} K^{j_1}_{i_1}\left\{\int\limits_0^1 dt \int\limits_0^t ds \left[\frac{1}{2} W^{j_2 j_3}_{i_2 i_3} + \left(1 - t^2\right) K^{j_2}_{i_2} K^{j_3}_{i_3} + \right.\right.$$
$$\left.\left. + \frac{s^2 - 1}{\ell^2}\delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\right]^{n-1} + \frac{(-1)^n 2^{2n-2}(n-1)!^2}{\ell^{2n-2}(2n-1)!}\delta^{j_2}_{i_2}\ldots\delta^{j_{2n-1}}_{i_{2n-1}}\right\} . \tag{3.79}$$

Because of the vanishing of the bulk Weyl tensor in the ACF case and considering the expansion of the extrinsic curvature up to cubic order (2.73), the lowest order approximation for the binomial expansion (Appendix C), gives

$$\tilde{\mathcal{L}}_{ct}(h,K,\mathcal{R}) = \frac{\sqrt{-h}}{8\pi G}\frac{(-1)^n\,\ell^{2n-2}}{2^{2n-2}(n-1)!^2}\delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]}\left(K^{j_1}_{(0)i_1}+\rho K^{j_1}_{(2)i_1}+\rho^2K^{j_1}_{(4)i_1}+\rho^3K^{j_1}_{(6)i_1}\right)\times$$
$$\left\{\int_0^1 dt\int_0^t ds\left[\left(s^2-t^2\right)^{n-1}M_0+\rho\Sigma_2+\rho^2\Sigma_4+\rho^3\Sigma_6\right]+\right.$$
$$\left.+\frac{(-1)^n\,\ell^{2n-2}\,(n-1)!^2}{(2n-1)!}M_0\right\}. \tag{3.80}$$

Below, we expand the series (3.80) using power-counting arguments, and evaluating the parametric integrals (Appendix D), we derive the relevant order of counterterms.

Starting from the 0-th order term, we obtain

$$\begin{aligned}\tilde{\mathcal{L}}_{ct(0)} &= \frac{\sqrt{-h}}{8\pi G}\frac{(-1)^n\,\ell^{2n-2}}{2^{2n-2}(n-1)!^2}\delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]}\int_0^1 dt\int_0^t ds\left[\left(s^2-t^2\right)^{n-1}+\right.\\ &+\left.\frac{(-1)^n\,2^{2n-2}\,(n-1)!^2}{(2n-1)!}\right]K^{j_1}_{(0)i_1}M_0\\ &= \frac{\sqrt{-h}}{8\pi G}\frac{2n-1}{\ell} = \frac{\sqrt{-h}}{8\pi G}\frac{2n-1}{2n}K_{(0)} = \mathcal{L}_{ct(0)}\,.\end{aligned} \tag{3.81}$$

Thus, we recover the leading order term of the standard counterterms as given in Eqs.(3.3) and (3.5).

Going to the second order counterterms, only terms linear in the curvature contribute. Thus,

$$\begin{aligned}\tilde{\mathcal{L}}_{ct(2)} &= \rho\frac{\sqrt{-h}}{8\pi G}\frac{(-1)^n\,\ell^{2n-2}}{2^{2n-2}(n-1)!^2}\delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]}\int_0^1 dt\int_0^t ds\left[\left(s^2-t^2\right)^{n-1}+\right.\\ &+\left.2(n-1)\left(1-t^2\right)\left(s^2-t^2\right)^{n-2}+\frac{(-1)^n\,2^{2n-2}\,(n-1)!^2}{(2n-1)!}\right]K^{j_1}_{(0)i_1}M_{2,0}\\ &= \frac{\sqrt{-h}}{8\pi G}\frac{2n-1}{2n-2}K_{(2)} = \mathcal{L}_{ct(2)}\,.\end{aligned} \tag{3.82}$$

As a consequence, the equivalence between the Kounterterms in odd-dimensional manifolds and the standard counterterm series are extended to term linear in the curvature. When expressed in terms of the intrinsic curvature, the $\tilde{\mathcal{L}}_{ct(2)}$ becomes

$$\tilde{\mathcal{L}}_{ct(2)} = \frac{\ell\sqrt{-h}}{8\pi G}\frac{1}{2(2n-2)}\mathcal{R}\,. \tag{3.83}$$

Summing up the contributions for the fourth order term, we find

$$\begin{aligned}
\tilde{\mathcal{L}}_{ct(4)} &= \rho^2 \frac{\sqrt{-h}}{8\pi G} \frac{(-1)^n \ell^{2n-2}}{2^{2n-2}(n-1)!^2} \delta^{[i_1 \dots i_{2n-1}]}_{[j_1 \dots j_{2n-1}]} \int_0^1 dt \int_0^t ds \left\{ K^{j_1}_{(0)i_1} \Sigma_4 + K^{j_1}_{(2)i_1} \Sigma_2 + \right. \\
&+ \left. \left[\left(s^2 - t^2\right)^{n-1} + \frac{(-1)^n 2^{2n-2} (n-1)!^2}{(2n-1)!} \right] K^{j_1}_{(4)i_1} M_0 \right\} \\
&= \frac{\sqrt{-h}}{8\pi G} \frac{1}{2n-2} \left[(2n-1) K_{(4)} - \frac{\ell}{2n-4} \delta^{[i_1 i_2]}_{[j_1 j_2]} K^{j_1}_{(2)i_1} K^{j_2}_{(2)i_2} \right] . \qquad (3.84)
\end{aligned}$$

Using the relation between the symmetric traces of the extrinsic curvature coefficients given in Eq.(2.79) on a $d = 2n$ boundary, the counterterm can be rewritten as

$$\tilde{\mathcal{L}}_{ct(4)} = \frac{\sqrt{-h}}{8\pi G} \frac{2n-1}{2n-4} K_{(4)} = \mathcal{L}_{ct(4)} \, , \qquad (3.85)$$

providing the correct result. Its form can be alternatively expressed as

$$\tilde{\mathcal{L}}_{ct(4)} = \frac{\sqrt{-h}}{8\pi G} \frac{\ell^3}{2(2n-2)^2 (2n-4)} \left(\mathcal{R}^i_j \mathcal{R}^j_i - \frac{n}{2(2n-1)} \mathcal{R}^2 \right) . \qquad (3.86)$$

For the sixth order, we get

$$\begin{aligned}
\tilde{\mathcal{L}}_{ct(6)} &= \rho^3 \frac{\sqrt{-h}}{8\pi G} \frac{(-1)^n \ell^{2n-2}}{2^{2n-2}(n-1)!^2} \delta^{[i_1 \dots i_{2n-1}]}_{[j_1 \dots j_{2n-1}]} \int_0^1 dt \int_0^t ds \left\{ K^{j_1}_{(0)i_1} \Sigma_6 + K^{j_1}_{(2)i_1} \Sigma_4 \right. \\
&+ \left. K^{j_1}_{(4)i_1} \Sigma_2 + \left[\left(s^2 - t^2\right)^{n-1} + \frac{(-1)^n 2^{2n-2} (n-1)!^2}{(2n-1)!} \right] K^{j_1}_{(6)i_1} M_0 \right\} \\
&= \frac{\sqrt{-h}}{8\pi G} \frac{1}{2n-2} \left[(2n-1) K_{(6)} - \frac{2\ell}{2n-4} \delta^{[i_1 i_2]}_{[j_1 j_2]} K^{j_1}_{(2)i_1} K^{j_2}_{(4)i_2} \right. \\
&+ \left. \frac{\ell^2}{(2n-4)(2n-6)} \delta^{[i_1 i_2 i_3]}_{[j_1 j_2 j_3]} K^{j_1}_{(2)i_1} K^{j_2}_{(2)i_2} K^{j_3}_{(2)i_3} \right] . \qquad (3.87)
\end{aligned}$$

Replacing the traces from the Eqs.(2.80) and (3.45), the counterterms become

$$\tilde{\mathcal{L}}_{ct(6)} = \frac{\sqrt{-h}}{8\pi G} \frac{2n-1}{2n-6} K_{(6)} = \mathcal{L}_{ct(6)} \, . \qquad (3.88)$$

This result makes manifest the proportionality between $\mathcal{L}_{ct(6)}$ and the trace $K_{(6)}$, with the correct coefficient as given by the Eq.(3.5).

In terms of the boundary curvature, it is given by

$$\tilde{\mathcal{L}}_{ct(6)} = \frac{\sqrt{-h}}{8\pi G}\frac{\ell^5}{(2n-2)^4(2n-6)}\left[\mathcal{R}^k_j\mathcal{R}^j_i\mathcal{R}^i_k - \frac{3n}{2(2n-1)}\mathcal{R}\mathcal{R}^i_j\mathcal{R}^j_i + \frac{n^2+2n-1}{16n^2}\mathcal{R}^3\right]. \tag{3.89}$$

Hence, even though B_{2n} was originally used to renormalize CS-AdS gravity, it is proved to match the standard counterterm series for Einstein-AdS gravity, at least up to the cubic order in the curvature, extending the equivalence between the Kounterterms and the HR to the odd-dimensional case.

3.3.3 Towards a general formula for the counterterms

Following the derivation in the even-dimensional case, we are seeking for the possibility to construct a closed formula for the counterterms, that would allow us to compute explicitly terms of arbitrary order $2p$. In order to do so, it is useful to apply the matrix representation. The counterterms in Eq.(3.79) are represented in this formulation as

$$\tilde{\mathcal{L}}_{\mathrm{ct}} = -\frac{\sqrt{-h}}{8\pi G}\left[\frac{1}{2^{2n-2}(n-1)!^2}\int_0^1 dt\int_0^t ds\left\langle \mathbb{K}\mathbb{I}\left(\left(1-s^2\right)\mathbb{I}^2-\left(1-t^2\right)\ell^2\mathbb{K}^2\right)^{n-1}\right\rangle - \langle\mathbb{K}\rangle\right] + \mathcal{O}(1)$$

where the bulk Weyl tensor has been dropped. Here we define the parametric integral

$$\begin{aligned}\mathcal{J}_k &= \int_0^1 dt\int_0^t ds\left(t^2-1\right)^k\left(t^2-s^2\right)^{n-1-k} \\ &= (-1)^k\frac{2^{2n-2k-3}k!\,(n-1-k)!^3}{n!(2n-2k-1)!},\end{aligned} \tag{3.90}$$

which allows us to simplify many of the expressions that will appear below. Applying the binomial expansion

$$\begin{aligned}&\left\langle \mathbb{K}\mathbb{I}\left(\left(1-s^2\right)\mathbb{I}^2-\left(1-t^2\right)\ell^2\mathbb{K}^2\right)^{n-1}\right\rangle \\ &= (n-1)!\sum_{k=0}^{n-1}\frac{\left(t^2-1\right)^k\left(t^2-s^2\right)^{n-1-k}}{k!\,(n-1-k)!}\left\langle \mathbb{I}^{2n-2k-1}\mathbb{K}\left(\ell^2\mathbb{K}^2-\mathbb{I}^2\right)^k\right\rangle,\end{aligned} \tag{3.91}$$

and plugging it into the counterterms integral, we obtain

$$\tilde{\mathcal{L}}_{\text{ct}} = -\frac{\sqrt{-h}}{8\pi G}\left[\frac{1}{n!(n-1)!}\sum_{k=1}^{n-1}\frac{(-1)^k(n-1-k)!^2}{2^{2k+1}(2n-2k-1)!}\left\langle \mathbb{I}^{2n-2k-1}\mathbb{K}\left(\ell^2\mathbb{K}^2-\mathbb{I}^2\right)^k\right\rangle \right.$$
$$\left. -\frac{2n-1}{2n}\langle\mathbb{K}\rangle\right]. \tag{3.92}$$

This expression can be simplified even further. Indeed, knowing that the terms appearing in the integral satisfy the relation

$$\left\langle \mathbb{I}^{2n-2k-1}\mathbb{K}\left(\ell^2\mathbb{K}^2-\mathbb{I}^2\right)^k\right\rangle = \sum_{l=0}^{k}\frac{(-1)^{k-l}\,k!\,\ell^{2l}}{l!\,(k-l)!}\left\langle \mathbb{I}^{2n-2l-1}\mathbb{K}^{2l+1}\right\rangle, \tag{3.93}$$

and evaluating Eq.(3.57), a new simplified formula for the counterterms arises

$$\tilde{\mathcal{L}}_{\text{ct}\,(2p)} = \frac{\sqrt{-h}}{8\pi G}\sum_{\{m_i\}}\frac{\ell^{m_0-1}C^{(2n)}_{m_1,\dots,m_p}}{m_1!\cdots m_p!}\left\langle \mathbb{K}^{m_1}_{(2)}\mathbb{K}^{m_2}_{(4)}\cdots\mathbb{K}^{m_p}_{(2p)}\right\rangle, \tag{3.94}$$

where the constraints in Eq.(3.55) are still valid and the coefficient acquires the value

$$C^{(2n)}_{m_1,\dots,m_p} = \sum_{k=1}^{n-1}\sum_{l=0}^{k}\frac{(-1)^l\,k!\,(n-1-k)!^2\,(2l+1)!\,(2n-m_0)!}{2^{2k+1}n!(n-1)!(2n-2k-1)!l!\,(k-l)!\,(2l+1-m_0)!}$$
$$-\frac{2n-1}{2n}\delta_{1m_p}. \tag{3.95}$$

A quick check shows that we reproduce correctly the counterterms of the last section, namely up to sixth order. Having this closed formula at hand, we can go beyond that order and check the compatibility between the Kounterterms and Holographic Renormalization. Applying the integral (3.94) for $p=4$, gives rise to

$$\tilde{\mathcal{L}}_{\text{ct}\,(8)} = -\frac{\sqrt{-\gamma}}{8\pi G}\frac{1}{d-2}\left[(d-1)\left\langle\mathbb{K}_{(8)}\right\rangle - \frac{2\ell}{d-4}\left\langle\mathbb{K}_{(2)}\mathbb{K}_{(6)}\right\rangle - \frac{\ell}{d-4}\left\langle\mathbb{K}^2_{(4)}\right\rangle\right.$$
$$\left. + \frac{3\ell^2}{(d-4)(d-6)}\left\langle\mathbb{K}^2_{(2)}\mathbb{K}_{(4)}\right\rangle - \frac{\ell^3}{(d-4)(d-6)(d-8)}\left\langle\mathbb{K}^4_{(2)}\right\rangle\right], \tag{3.96}$$

where $d=2n$.

This result is the same as in the even-dimensional case and an impossibility to be written uniquely in terms of $\langle K_{(8)}\rangle$ confirms the mismatch of the two regularizing schemes when we go to quartic order in the curvature, in the odd-dimensional case as well.

3.4 Comments on the Kounterterms

Even though the n-th Chern form in even dimensions and transgression form in odd dimensions are quantities that correspond to distinct geometrical structures, it was proved that they are in agreement with the standard counterterms at least up to sixth order in the expansion. This feature led us to construct a closed recursive formula that calculates the counterterms to all orders and depends explicitly on the symmetric traces of the various terms of the extrinsic curvature expansion. I expect that this formula will match the standard counterterms to all orders.

The mismatch that arises at the 8th order can possibly be explained as an incompleteness of the ACF condition. Indeed, for the derivation of the ACF condition considers only the leading order term of the extrinsic curvature expansion whereas for the derivation of the Kounterterms the next to the leading order term is becoming relevant, which would induce new constraints among the extrinsic curvature components and possibly lead to proportionality of Kounterterms $\tilde{\mathcal{L}}_{ct(2p)}$ to $\langle K_{2p} \rangle$. Taking into account the linear term in the curvature corresponds to a modified asymptotic behavior of the Weyl tensor, leading the ACF approximation not to be sufficient after a specific order.

One evidence that supports this claim is that conserved charges obtained from $\mathcal{L}_{ct}$ or $\tilde{\mathcal{L}}_{ct}$ are the same,i.e., a difference between them is not observable. Similar arguments hold for other observable quantities, such as holographic anomalies and Entanglement Entropy.

Chapter 4

Conformal Gravity

4.1 Introduction

EH gravity, since its introduction from A. Einstein in 1915, has been the main paradigm for the description of gravity and constitutes one of the main cornerstones of Modern Physics. In the next decades, its compatibility with quantum mechanics was put in question, and a quest for a quantum theory of gravity began. It was proved later Ref.[19], that GR is a two-loop divergent theory and does not exhibit the desirable features of quantum gravity.

String theory, on the other hand. is a finite theory and in its low-energy limit arise non-linear terms in the curvature, indicating that higher-order extensions of GR may be plausible. Indeed, higher-curvature gravity theories proved to be renormalizable [20], [21] provided a non-divergent ultraviolet (UV) behavior.

Due to the fact that renormalizability is considered a prerequisite for quantum gravity, higher-derivative theories of gravity were extensively studied during the last decades. Some candidates are massive gravity theories, $F(R)$, etc. For instance, in three dimensions we meet NMG [22] and TMG [23]. In four dimensions, there are two main examples: i) CG and ii) CrG [24]. The case of CrG be studied further in the next chapter and here we focus on CG.

CG is a special case in the class of higher-derivative theories of gravity. It is a quadratic-curvature theory of gravity that is renormalizable Ref.[25], but the presence of higher-derivatives terms in the EOM lead to ghosts (modes of negative norm), contrary to EH gravity. Its action acquires the form

$$\begin{aligned} I_{CG} &= \alpha_{CG} \int_M d^4x \sqrt{-g} W^{\alpha\beta\mu\nu} W_{\alpha\beta\mu\nu} \\ &= \frac{\alpha_{CG}}{4} \int_M d^4x \sqrt{-g} \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} W^{\mu_1\mu_2}_{\nu_1\nu_2} W^{\mu_3\mu_4}_{\nu_3\nu_4} , \end{aligned} \tag{4.1}$$

where $W^{\alpha\beta}_{\mu\nu}$ is the Weyl tensor (2.18).

CG in four dimensions is invariant under local Weyl rescalings of the metric $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$, namely a transformation that preserves the angles but not the distances. Furthermore, the coupling constant, α_{CG}, is a positive dimensionless parameter, making the theory scale-invariant.

The theory is a recurrent topic in the literature for various reasons. Phenomenologically, earlier works [26]–[29] suggest CG as an alternative candidate to dark matter, due to its ability to describe the galactic rotation curves. At a more fundamental level, the action (4.1) emerges from twistor string theory [30].

Recently, Maldacena showed in Ref. [31] the equivalence between EH gravity with a cosmological constant and four-dimensional CG at tree level, considering specific Neumann boundary conditions. Based on the concept of topological regularization, previously discussed in section 2.3, it was shown in [32] that the regularized action for Einstein-AdS gravity is on-shell equivalent to the action of CG for the specific value of the coupling constant $\alpha_{CG} = \ell^2/64\pi G$.

In what follows we try to prove the equivalence starting from the CG action based on the separability of the curvature into two distinct parts: Einstein and NE, respectively. The reason behind this decomposition is the fact that CG possesses a broad class of solutions, in which Einstein spaces constitute a particular subset. With this in mind, we isolate the Einstein part of the CG action from the higher-curvature contributions and provide an explicit proof of the equivalence between CG and Einstein gravity.

4.2 Derivation of the field equations and surface terms

4.2.1 Field equations

Starting from the CG action (4.1) in a four-dimensional manifold M, we proceed firstly with the variation of it. The variation consist of two terms,

$$\begin{aligned} \delta I_{CG} &= \frac{\alpha_{CG}}{4} \int_M d^4x \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} \left[\delta\sqrt{-g} W^{\mu_1\mu_2}_{\nu_1\nu_2} W^{\mu_3\mu_4}_{\nu_3\nu_4} + 2\sqrt{-g}\delta W^{\mu_1\mu_2}_{\nu_1\nu_2} W^{\mu_3\mu_4}_{\nu_3\nu_4}\right] \\ &= \frac{\alpha_{CG}}{4} \int_M d^4x \sqrt{-g} \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} \left[\frac{1}{2} W^{\mu_1\mu_2}_{\nu_1\nu_2} W^{\mu_3\mu_4}_{\nu_3\nu_4} \left(g^{-1}\delta g\right) + 2\delta W^{\mu_1\mu_2}_{\nu_1\nu_2} W^{\mu_3\mu_4}_{\nu_3\nu_4}\right], \end{aligned} \tag{4.2}$$

where $g_{\mu\nu}$ is the bulk metric. To simplify it, we identify particular terms as

$$I_1 = \frac{1}{2}\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} W^{\mu_1\mu_2}_{\nu_1\nu_2} W^{\mu_3\mu_4}_{\nu_3\nu_4} \left(g^{-1}\delta g\right), \tag{4.3}$$

$$I_2 = 2\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} \delta W^{\mu_1\mu_2}_{\nu_1\nu_2} W^{\mu_3\mu_4}_{\nu_3\nu_4}. \tag{4.4}$$

The second term can be equivalently written as

$$\begin{aligned} I_2 &= \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}\delta W^{\mu_1\mu_2}_{\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4} \\ &= \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}\delta\left(g^{\mu_2\kappa}W^{\mu_1}_{\kappa\nu_1\nu_2}\right)W^{\mu_3\mu_4}_{\nu_3\nu_4} \\ &= \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}\left(\delta g^{\mu_2\kappa}W^{\mu_1}_{\kappa\nu_1\nu_2}+g^{\mu_2\kappa}\delta W^{\mu_1}_{\kappa\nu_1\nu_2}\right)W^{\mu_3\mu_4}_{\nu_3\nu_4}\,. \end{aligned}$$

After some algebraic manipulation, the first part of I_2 reads

$$\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}\delta g^{\mu_2\kappa}W^{\mu_1}_{\kappa\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4} = -\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}\left(g^{-1}\delta g\right)^{\mu_2}_{\nu_5}\delta^{\nu_5}_{\mu_5}W^{\mu_1\mu_5}_{\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4}\,.$$

For the second part, and taking into account the definition of the Weyl in Eq.(2.18), we get

$$\begin{aligned} &\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}g^{\mu_2\kappa}\delta W^{\mu_1}_{\kappa\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4} = \\ &= \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}g^{\mu_2\kappa}\delta\left[R^{\mu_1}_{\kappa\nu_1\nu_2}-\left(S^{\mu_1}_{\nu_1}g_{\kappa\nu_2}-S^{\mu_1}_{\nu_2}g_{\kappa\nu_1}-S_{\kappa\nu_1}\delta^{\mu_1}_{\nu_2}+S_{\kappa\nu_2}\delta^{\mu_1}_{\nu_1}\right)\right]W^{\mu_3\mu_4}_{\nu_3\nu_4} \\ &= \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}\left[g^{\mu_2\kappa}\delta R^{\mu_1}_{\kappa\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4}-2g^{\mu_2\kappa}\delta\left(S^{\mu_1}_{\nu_1}g_{\kappa\nu_2}\right)W^{\mu_3\mu_4}_{\nu_3\nu_4}\right]-2\delta^{[\nu_1\nu_2\nu_3]}_{[\mu_1\mu_2\mu_3]}g^{\mu_1\kappa}\delta S_{\kappa\nu_1}W^{\mu_2\mu_3}_{\nu_2\nu_3}\,, \end{aligned}$$

where the last term vanishes due to the appearance of traces of the Weyl. Here, S^{μ}_{ν} is the Schouten tensor of the bulk (2.19), which in four dimensions reads

$$S^{\alpha}_{\mu} = \frac{1}{2}(R^{\alpha}_{\mu}-\frac{1}{6}R\delta^{\alpha}_{\mu})\,. \tag{4.5}$$

The second term involving $\delta S^{\mu_1}_{\nu_1}$ is analyzed as follows,

$$\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}g^{\mu_2\kappa}\delta\left(S^{\mu_1}_{\nu_1}g_{\kappa\nu_2}\right)W^{\mu_3\mu_4}_{\nu_3\nu_4} = \delta^{[\nu_1\nu_2\nu_3]}_{[\mu_1\mu_2\mu_3]}\delta S^{\mu_1}_{\nu_1}W^{\mu_2\mu_3}_{\nu_2\nu_3}+\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}S^{\mu_1}_{\nu_1}W^{\mu_2\mu_3}_{\nu_2\nu_3}\left(g^{-1}\delta g\right)^{\mu_4}_{\nu_4}\,,$$

receiving contributions only from the last term. Summing up all the contributions in Eq.(4.2), we get

$$\begin{aligned} \delta I_{CG} = \frac{\alpha_{CG}}{4}\int_M d^4x\sqrt{-g}\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}\Big[&\frac{1}{2}\delta^{\nu_5}_{\mu_5}W^{\mu_1\mu_2}_{\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4}\left(g^{-1}\delta g\right)^{\mu_5}_{\nu_5}-2\delta^{\nu_5}_{\mu_5}W^{\mu_1\mu_5}_{\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4}\left(g^{-1}\delta g\right)^{\mu_2}_{\nu_5} \\ &-4S^{\mu_1}_{\nu_1}W^{\mu_2\mu_3}_{\nu_2\nu_3}\left(g^{-1}\delta g\right)^{\mu_4}_{\nu_4}+2g^{\mu_2\kappa}\delta R^{\mu_1}_{\kappa\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4}\Big]\,. \end{aligned}$$

The first line in the variation can be simplified using the expansion of the antisymmetric delta as

$$\frac{1}{2}\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}\delta^{\nu_5}_{\mu_5}\left[W^{\mu_1\mu_2}_{\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4}\left(g^{-1}\delta g\right)^{\mu_5}_{\nu_5}-4W^{\mu_1\mu_5}_{\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4}\left(g^{-1}\delta g\right)^{\mu_2}_{\nu_5}\right]=$$
$$=\frac{1}{2}\delta^{[\nu_1\nu_2\nu_3\nu_4\nu_5]}_{[\mu_1\mu_2\mu_3\mu_4\mu_5]}W^{\mu_1\mu_2}_{\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4}\left(g^{-1}\delta g\right)^{\mu_5}_{\nu_5}. \tag{4.6}$$

In $D=4$ this term vanishes, with the surviving contributions in the variation of the CG action to be

$$\delta I_{CG}=\frac{\alpha_{CG}}{2}\int_M d^4x\sqrt{-g}\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}\left[g^{\mu_2\kappa}\delta R^{\mu_1}_{\kappa\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4}-2S^{\mu_1}_{\nu_1}W^{\mu_2\mu_3}_{\nu_2\nu_3}\left(g^{-1}\delta g\right)^{\mu_4}_{\nu_4}\right] \tag{4.7}$$

Once more, we rename the two terms as

$$\begin{aligned} I_3 &= \int\limits_M d^4x\sqrt{-g}\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}g^{\mu_2\kappa}\delta R^{\mu_1}_{\kappa\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4}, \\ I_4 &= 2\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}S^{\mu_1}_{\nu_1}W^{\mu_2\mu_3}_{\nu_2\nu_3}\left(g^{-1}\delta g\right)^{\mu_4}_{\nu_4}, \end{aligned}$$

in order to simplify the derivation below. Considering the variation of the Riemann tensor

$$\begin{aligned} \delta R^{\mu_1}_{\kappa\nu_1\nu_2} &= \nabla_{\nu_1}\delta\Gamma^{\mu_1}_{\kappa\nu_2}-\nabla_{\nu_2}\delta\Gamma^{\mu_1}_{\kappa\nu_1}, \\ \delta\Gamma^{\mu_1}_{\kappa\nu_2} &= \frac{1}{2}g^{\mu_1\sigma}\left(\nabla_\kappa\delta g_{\sigma\nu_2}+\nabla_{\nu_2}\delta g_{\kappa\sigma}-\nabla_\sigma\delta g_{\kappa\nu_2}\right), \end{aligned}$$

the contribution coming from I_3 is expressed as

$$\begin{aligned} I_3 &= \int_M d^4x\sqrt{-g}\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}g^{\mu_2\kappa}\delta R^{\mu_1}_{\kappa\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4} \\ &= 2\int_M d^4x\sqrt{-g}\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}g^{\mu_2\kappa}\nabla_{\nu_1}\delta\Gamma^{\mu_1}_{\kappa\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4} \\ &= -2\int_M d^4x\sqrt{-g}\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}g^{\mu_2\kappa}\delta\Gamma^{\mu_1}_{\kappa\nu_2}\nabla_{\nu_1}W^{\mu_3\mu_4}_{\nu_3\nu_4} \\ &+ 2\int_{\partial M} d^3x\sqrt{-h}\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}n_{\nu_1}g^{\mu_2\kappa}\delta\Gamma^{\mu_1}_{\kappa\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4}, \end{aligned} \tag{4.8}$$

where integration by parts was applied to pass from second to third line. Here, $\nabla_\mu=\nabla_\mu(\Gamma)$ is the affine covariant derivative in the bulk.

The bulk term contributes to the EOM and, when expanded, acquires the form

$$
\begin{aligned}
&- \int_M d^4x \sqrt{-g} \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} g^{\mu_2\kappa} g^{\mu_1\sigma} \left(\nabla_\kappa \delta g_{\sigma\nu_2} + \nabla_{\nu_2} \delta g_{\kappa\sigma} - \nabla_\sigma \delta g_{\kappa\nu_2} \right) \nabla_{\nu_1} W^{\mu_3\mu_4}_{\nu_3\nu_4} = \\
&= 2 \int_M d^4x \sqrt{-g} \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} \nabla^{\mu_1} \left(g^{-1}\delta g \right)^{\mu_2}_{\nu_2} \nabla_{\nu_1} W^{\mu_3\mu_4}_{\nu_3\nu_4} \\
&= -2\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} \left(\int_M d^4x \sqrt{-g} \nabla^{\mu_1} \nabla_{\nu_1} W^{\mu_2\mu_3}_{\nu_2\nu_3} - \int_{\partial M} d^3x \sqrt{-h} n^{\mu_1} \nabla_{\nu_1} W^{\mu_2\mu_3}_{\nu_2\nu_3} \right) \left(g^{-1}\delta g \right)^{\mu_4}_{\nu_4},
\end{aligned}
$$

where the middle term of the first line vanishes due to antisymmetry in the indices $(\kappa\sigma)$. Furthermore, for the I_4 term we obtain

$$
2\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} S^{\mu_1}_{\nu_1} W^{\mu_2\mu_3}_{\nu_2\nu_3} \left(g^{-1}\delta g \right)^{\mu_4}_{\nu_4} = \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} R^{\mu_1}_{\nu_1} W^{\mu_2\mu_3}_{\nu_2\nu_3} \left(g^{-1}\delta g \right)^{\mu_4}_{\nu_4} .
$$

Summing all the contributions, the variation of the action adopts the form

$$
\begin{aligned}
\delta I_{CG} = \alpha_{CG} \int_M d^4x \sqrt{-g} B^\nu_\mu \left(g^{-1}\delta g \right)^\mu_\nu + \alpha_{CG} \int_{\partial M} d^3x \sqrt{-h} \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} \Big[n_{\nu_1} \delta\Gamma^{\mu_1}_{\kappa\nu_2} g^{\mu_2\kappa} W^{\mu_3\mu_4}_{\nu_3\nu_4} + \\
+ n^{\mu_1} \nabla_{\nu_1} W^{\mu_2\mu_3}_{\nu_2\nu_3} \left(g^{-1}\delta g \right)^{\mu_4}_{\nu_4} \Big] , \qquad (4.9)
\end{aligned}
$$

where n_μ is normal to the boundary, and B^ν_μ is written as

$$
\begin{aligned}
B^\nu_\mu &= -\delta^{[\nu\nu_1\nu_2\nu_3]}_{[\mu\mu_1\mu_2\mu_3]} \left(\nabla^{\mu_1} \nabla_{\nu_1} W^{\mu_2\mu_3}_{\nu_2\nu_3} + \frac{1}{2} R^{\mu_1}_{\nu_1} W^{\mu_2\mu_3}_{\nu_2\nu_3} \right) \\
&= -4 \left(\nabla^\alpha \nabla_\beta W^{\beta\nu}_{\alpha\mu} + \frac{1}{2} R^\alpha_\beta W^{\beta\nu}_{\alpha\mu} \right) . \qquad (4.10)
\end{aligned}
$$

The result is a four-derivative, traceless and covariantly conserved tensor in four dimensions, called the Bach tensor.

The EOM of CG, which are defined by the vanishing of the bulk term in (4.9), are satisfied by Bach flat solutions,i.e., $B_{\mu\nu} = 0$. This class of solutions consists of two main families of spacetimes: i) locally conformal Einstein spaces, and ii) self-dual or anti-self-dual spaces. Einstein spacetimes are Bach flat as well, as it can be confirmed by plugging in the relation (2.17) into (4.10), and they form a trivial subset of the whole family of solutions.

The presence of a higher-derivative tensor in the field equations leads to both massless and massive propagating modes with opposite norms. This means that, except from the normal ones, there are ghosts as well, namely, modes with negative kinetic energy. Recently, there have been works [33] on realizations of the theory where the ghosts are absent and unitarity is restored.

Another interesting feature arising in CG is associated to the presence of massive modes, is known as the phenomenon of partial masslessness [34], [35]. This phenomenon occurs when the scalar component of the massive mode is absent and the only non-vanishing contributions of the decomposition are the ones corresponding to spin 2 and spin 1 modes.

4.2.2 Surface terms

In this subsection we proceed with a more detailed analysis of the surface terms derived from the first variation of the CG action

$$\delta I_{surf} = \alpha_{CG} \int_{\partial M} d^3x \sqrt{-h} \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} \left[n_{\nu_1} \delta\Gamma^{\mu_1}_{\kappa\nu_2} g^{\mu_2\kappa} W^{\mu_3\mu_4}_{\nu_3\nu_4} + \right. \\ \left. + n^{\mu_1} \nabla_{\nu_1} W^{\mu_2\mu_3}_{\nu_2\nu_3} \left(g^{-1}\delta g \right)^{\mu_4}_{\nu_4} \right] . \tag{4.11}$$

In what follows, we are assuming a radial foliation of the spacetime of the form given in Eq.(2.23), while we decompose the surface term in two parts,

$$\delta I_1 = \int\limits_{\partial M} d^3x \sqrt{-h} \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} n_{\nu_1} \delta\Gamma^{\mu_1}_{\kappa\nu_2} g^{\mu_2\kappa} W^{\mu_3\mu_4}_{\nu_3\nu_4} , \tag{4.12}$$

$$\delta I_2 = \int\limits_{\partial M} d^3x \sqrt{-h} \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} n^{\mu_1} \nabla_{\nu_1} W^{\mu_2\mu_3}_{\nu_2\nu_3} \left(g^{-1}\delta g \right)^{\mu_4}_{\nu_4} . \tag{4.13}$$

Writing the first surface term in the radial foliation, we get

$$\begin{aligned} \delta I_1 =& \int_{\partial M} d^3x \sqrt{-h} \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} n_{\nu_1} \delta\Gamma^{\mu_1}_{\kappa\nu_2} g^{\mu_2\kappa} W^{\mu_3\mu_4}_{\nu_3\nu_4} \\ =& \int_{\partial M} d^3x \sqrt{-h} N \delta^{[i_1i_2i_3]}_{[j_1j_2j_3]} \left(\delta\Gamma^{\rho}_{\kappa i_1} g^{j_1\kappa} W^{j_2j_3}_{i_2i_3} - \delta\Gamma^{j_1}_{\kappa i_1} g^{\rho\kappa} W^{j_2j_3}_{i_2i_3} - 2\delta\Gamma^{j_1}_{\kappa i_1} g^{j_2\kappa} W^{j_3\rho}_{i_2i_3} \right) \\ =& \int_{\partial M} d^3x \sqrt{-h} N \delta^{[i_1i_2i_3]}_{[j_1j_2j_3]} \left(\frac{1}{N} \delta K_{\kappa i_1} h^{j_1\kappa} W^{j_2j_3}_{i_2i_3} + \frac{1}{N} \delta K^{j_1}_{i_1} W^{j_2j_3}_{i_2i_3} - 2\delta\Gamma^{j_1}_{\kappa i_1} h^{j_2\kappa} W^{j_3\rho}_{i_2i_3} \right) \\ =& \int_{\partial M} d^3x \sqrt{-h} N \delta^{[i_1i_2i_3]}_{[j_1j_2j_3]} \left[\frac{1}{N} \left(2\delta K^{j_1}_{i_1} + K^{m}_{i_1} \left(h^{-1}\delta h \right)^{j_1}_{m} \right) W^{j_2j_3}_{i_2i_3} - 2\delta\Gamma^{j_1}_{\kappa i_1} h^{j_2\kappa} W^{j_3\rho}_{i_2i_3} \right] , \end{aligned}$$

where the decomposition of the Christoffel symbol in the radial foliation (2.25) was taken into account. The third term can be further expanded as

$$
\begin{aligned}
& 2N\int_{\partial M} d^3x\sqrt{-h}\,\delta^{[i_1i_2i_3]}_{[j_1j_2j_3]}\delta\Gamma^{j_1}_{\kappa i_1}h^{j_2\kappa}W^{j_3\rho}_{i_2i_3} = \\
= \;& 2N\int_{\partial M} d^3x\sqrt{-h}\,\delta^{[i_1i_2i_3]}_{[j_1j_2j_3]}\frac{1}{2}h^{j_1m}h^{j_2\kappa}\left(D_\kappa\delta h_{mi_1}+D_{i_1}\delta h_{\kappa m}-D_m\delta h_{\kappa i_1}\right)W^{j_3\rho}_{i_2i_3} \\
= \;& 2N\int_{\partial M} d^3x\sqrt{-h}\,\delta^{[i_1i_2i_3]}_{[j_1j_2j_3]}D^{j_2}\left(h^{-1}\delta h\right)^{j_1}_{i_1}W^{j_3\rho}_{i_2i_3},
\end{aligned}
$$

where $D_i = D_i\left(\Gamma^j_{kl}(h)\right)$ is the covariant derivative of the boundary. As we have seen before, the middle term in the second line vanishes due to antisymmetry. Thus, we get

$$
\begin{aligned}
\delta I_1 = \int_{\partial M} d^3x\sqrt{-h}\,\delta^{[i_1i_2i_3]}_{[j_1j_2j_3]}\Big\{&\left[2\delta K^{j_1}_{i_1}+K^m_{i_1}\left(h^{-1}\delta h\right)^{j_1}_m\right]W^{j_2j_3}_{i_2i_3}+ \\
&+2ND^{j_2}W^{j_3\rho}_{i_2i_3}\left(h^{-1}\delta h\right)^{j_1}_{i_1}\Big\}\,.
\end{aligned} \tag{4.14}
$$

Going to the second surface term of the variation of the action, one gets

$$
\begin{aligned}
\delta I_2 &= \int_{\partial M} d^3x\sqrt{-h}\,\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}n^{\mu_1}\nabla_{\nu_1}W^{\mu_2\mu_3}_{\nu_2\nu_3}\left(g^{-1}\delta g\right)^{\mu_4}_{\nu_4} \\
&= \int_{\partial M} d^3x\sqrt{-h}\,\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}n^{\mu_1}\nabla_{\nu_1}\left(R^{\mu_2\mu_3}_{\nu_2\nu_3}-4S^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_3}\right)\left(g^{-1}\delta g\right)^{\mu_4}_{\nu_4} \\
&= -4\int_{\partial M} d^3x\sqrt{-h}\,\delta^{[\nu_1\nu_2\nu_3]}_{[\mu_1\mu_2\mu_3]}n^{\mu_1}\nabla_{\nu_1}S^{\mu_2}_{\nu_2}\left(g^{-1}\delta g\right)^{\mu_3}_{\nu_3}\,.
\end{aligned}
$$

To pass from the second to the third line, the Bianchi identity was taken into account. Actually, this term is proportional to the Cotton tensor of the bulk, which is defined as

$$
C^\alpha_{\mu\nu} = \nabla_\mu S^\alpha_\nu - \nabla_\nu S^\alpha_\mu\,. \tag{4.15}
$$

Not all the components of the Cotton tensor contribute to δI_2. Applying the radial foliation, we find

$$
\begin{aligned}
\delta I_2 &= \int_{\partial M} d^3x\sqrt{-h}\,\delta^{[\nu_1\nu_2\nu_3]}_{[\mu_1\mu_2\mu_3]}n^{\mu_1}\nabla_{\nu_1}S^{\mu_2}_{\nu_2}\left(g^{-1}\delta g\right)^{\mu_3}_{\nu_3} \\
&= \frac{1}{N}\int_{\partial M} d^3x\sqrt{-h}\,\delta^{[i_1i_2]}_{[j_1j_2]}\left[\nabla_\rho S^{j_1}_{i_1}\left(h^{-1}\delta h\right)^{j_2}_{i_2}-\nabla_{i_1}S^{j_1}_\rho\left(h^{-1}\delta h\right)^{j_2}_{i_2}+\nabla_{i_1}S^{j_2}_{i_2}\left(h^{-1}\delta h\right)^{j_3}_\rho\right] \\
&= \frac{1}{N}\int_{\partial M} d^3x\sqrt{-h}\,\delta^{[i_1i_2]}_{[j_1j_2]}\left(\nabla_\rho S^{j_1}_{i_1}-\nabla_{i_1}S^{j_1}_\rho\right)\left(h^{-1}\delta h\right)^{j_2}_{i_2}\,.
\end{aligned}
$$

Thus, summing up all the contributions, the total surface term of the CG action reads

$$\delta I_{surf} = \alpha_{CG} \int_{\partial M} d^3x\sqrt{-h} \left\{ \delta^{[i_1 i_2 i_3]}_{[j_1 j_2 j_3]} \left[\left(2\delta K^{j_1}_{i_1} + K^{m}_{i_1} \left(h^{-1}\delta h\right)^{j_1}_{m} \right) W^{j_2 j_3}_{i_2 i_3} + 2ND^{j_1} W^{j_2 \rho}_{i_1 i_2} \left(h^{-1}\delta h\right)^{j_3}_{i_3} \right] - \frac{4}{N} \delta^{[i_1 i_2]}_{[j_1 j_2]} \left(\nabla_\rho S^{j_1}_{i_1} - \nabla_{i_1} S^{j_1}_{\rho} \right) \left(h^{-1}\delta h\right)^{j_2}_{i_2} \right\} , \quad (4.16)$$

or in terms of the Cotton tensor

$$\delta I_{surf} = \alpha_{CG} \int_{\partial M} d^3x\sqrt{-h} \left\{ \delta^{[i_1 i_2 i_3]}_{[j_1 j_2 j_3]} \left[\left(2\delta K^{j_1}_{i_1} + K^{m}_{i_1} \left(h^{-1}\delta h\right)^{j_1}_{m} \right) W^{j_2 j_3}_{i_2 i_3} + 2ND^{j_1} W^{j_2 \rho}_{i_1 i_2} \left(h^{-1}\delta h\right)^{j_3}_{i_3} \right] - \frac{4}{N} \delta^{[i_1 i_2]}_{[j_1 j_2]} C^{j_1}_{\rho i_1} \left(h^{-1}\delta h\right)^{j_2}_{i_2} \right\} . \quad (4.17)$$

This derivation of the surface terms for a Weyl squared action will be extremely useful in the study of the equivalence between Conformal and Einstein-AdS gravity below.

4.3 Topological Regularization and the variational problem in 4D Einstein-AdS gravity

One of the key steps towards the derivation of the equivalence between the two theories is the introduction of the topological regularization. In the last chapter, we showed that the Kounterterms provides a successful prescription for the cancellation of divergences in Einstein gravity and equivalent to the Holographic Renormalization, at least up to eleven dimensions. For even-dimensional bulk, the addition of the n-th Chern form is equivalent to the addition of a topological invariant of the Euler class for the corresponding dimension (3.28). Hence, the addition of a topological invariant with a fixed coupling constant on top of the Einstein Hilbert action renders the action finite, introducing in this way, topological regularization. The explicit form of the renormalized action is given in Eq.(3.29).

The first hint towards this new scheme was given in Ref.[32], in the case of four dimensions. The corresponding topological invariant is the GB term and the renormalized action acquires the form

$$I^{(E)}_{ren} = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g} \left(R - 2\Lambda + \frac{\ell^2}{16} \delta^{[\nu_1 \nu_2 \nu_3 \nu_4]}_{[\mu_1 \mu_2 \mu_3 \mu_4]} R^{\mu_1 \mu_2}_{\nu_1 \nu_2} R^{\mu_3 \mu_4}_{\nu_3 \nu_4} \right) , \quad (4.18)$$

where, from now on, the index (E) indicates quantities associated to EH gravity. In four dimensions the Euler theorem reads

$$\int_M d^4 x GB = 32\pi^2 \chi(M) + \int_{\partial M} d^3 x B_3 \,,$$

where

$$B_3 = 4\sqrt{-h}\delta^{[i_1 i_2 i_3]}_{[j_1 j_2 j_3]} K^{j_1}_{i_1} \left(\frac{1}{2}\mathcal{R}^{j_2 j_3}_{i_2 i_3} - \frac{1}{3} K^{j_2}_{i_2} K^{j_3}_{i_3} \right) .$$

The boundary dynamics cannot distinguish between the Chern form and the GB, as they are locally equivalent. Hence, the GB term can be replaced by the second Chern form, expressed in terms of intrinsic and extrinsic curvatures of the boundary [7],

$$I^{(E)}_{ren} = \frac{1}{16\pi G}\int_M d^4 x \sqrt{-g}\,(R - 2\Lambda) + \frac{\ell^2}{16\pi G}\int_{\partial M} d^3 x \sqrt{-h}\delta^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]} K^{i_1}_{j_1} \left(\frac{1}{2}\mathcal{R}^{i_2 i_3}_{j_2 j_3} - \frac{1}{3} K^{i_2}_{j_2} K^{i_3}_{j_3} \right) . \quad (4.19)$$

Of course, at the level of the Euclidean action there is a finite shift proportional to the Euler characteristic. Here, h_{ij} is the three-dimensional boundary metric.

By adding and subtracting the Gibbons-Hawking term we bring the action $I^{(E)}_{ren}$ in the form (3.2), where the counterterms become

$$\tilde{\mathcal{L}}_{ct} = \frac{\ell^2}{16\pi G}\sqrt{-h}\delta^{[i_1 i_2 i_3]}_{[j_1 j_2 j_3]} K^{j_1}_{i_1} \left(\frac{1}{2}\,\mathcal{R}^{j_2 j_3}_{i_2 i_3}(h) - \frac{1}{3}\,K^{j_2}_{i_2} K^{j_3}_{i_3} + \frac{1}{\ell^2}\,\delta^{j_2}_{i_2}\delta^{j_3}_{i_3} \right) .$$

As we saw in chapter 1, the key ingredient is the fact that the asymptotic solution of the Einstein equations, previously seen in subsection 2.2.1, induces the expansion of the extrinsic curvature in terms of intrinsic quantities of the boundary given in (2.26). Plugging this expansion into $\tilde{\mathcal{L}}_{ct}$, we pass from the extrinsic to the intrinsic counterterms which, after some algebraic manipulation, generates the standard counterterms

$$\mathcal{L}_{ct} = \frac{1}{8\pi G}\sqrt{-h}\left(\frac{2}{\ell} + \frac{\ell}{2}\mathcal{R}(h) \right) + \dots \,,$$

previously seen in Ref.[36]. This derivation presents a concrete realization of the discussion of Chapter 1, in the case of four dimensions, and expresses the equivalence between Topological Regularization and HR.

The addition of a bulk topological invariant implies that the renormalized action (4.18) obtains the form [32]

$$I_{MM} = \frac{\ell^2}{256\pi G}\int_{\mathcal{M}} d^4 x \sqrt{-g}\delta^{[\kappa\lambda\mu\nu]}_{[\alpha\beta\gamma\delta]} (R^{\alpha\beta}_{\kappa\lambda} + \frac{1}{\ell^2}\,\delta^{[\alpha\beta]}_{[\kappa\lambda]})(R^{\gamma\delta}_{\mu\nu} + \frac{1}{\ell^2}\,\delta^{[\gamma\delta]}_{[\mu\nu]}) \,,$$

what is the MacDowell-Mansouri action of the AdS group. This remarkable feature of topological regularization plays a significant role for the derivation below.

Furthermore, we realize that the Weyl tensor evaluated for Einstein spaces ($R_{\mu\nu} = -(3/\ell^2)g_{\mu\nu}$) coincides with the AdS curvature,

$$W^{\mu_1\mu_2}_{(E)\nu_1\nu_2} = R^{\mu_1\mu_2}_{\nu_1\nu_2} + \frac{1}{\ell^2}\delta^{[\mu_1\mu_2]}_{[\nu_1\nu_2]} \,. \tag{4.20}$$

This relation can be easily derived by plugging in the on-shell form of the Schouten tensor (2.20) into the Weyl (2.18). Thus, in Einstein gravity the AdS curvature is replaced by the on-shell Weyl tensor and the renormalized AdS action can be written as

$$I_{MM} = \frac{\ell^2}{256\pi G}\int_M d^4x\sqrt{-g}\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}W^{\mu_1\mu_2}_{(E)\nu_1\nu_2}W^{\mu_3\mu_4}_{(E)\nu_3\nu_4} \,. \tag{4.21}$$

This is actually the CG action when evaluated in Einstein spacetimes.

The Eq.(4.21), shows the equivalence between the Einstein-AdS gravity and CG for Einstein spacetimes, starting from the EH gravity side. In the next subsection, we will follow the inverse path, namely trying to obtain the equivalence starting from the CG side. Moreover, the identification of the renormalized action as Weyl squared has a profound geometrical origin, as it expresses the renormalized volume for AdS spacetimes, given in Ref.[37], and provides a hint for the connection between the renormalized action and the renormalized volume.

Finally, due to the quadratic in Weyl form of the renormalized AdS action, its variation is given by the surface term (4.11), where the Weyl tensor is now replaced by its on-shell form (4.20). In this case, the derivative term vanishes due to the Bianchi identity and the remaining contribution reads

$$\begin{aligned}\delta I_{MM} &= \frac{\ell^2}{64\pi G}\int_{\partial M} d^3x\sqrt{-h}\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}n_{\nu_1}\delta\Gamma^{\mu_1}_{\kappa\nu_2}g^{\mu_2}(R^{\mu_3\mu_4}_{\nu_3\nu_4} + \frac{1}{\ell^2}\delta^{[\mu_3\mu_4]}_{[\nu_3\nu_4]}) \\ &= \frac{\ell^2}{64\pi G}\int_{\partial M} d^3x\sqrt{-h}\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}n_{\nu_1}\delta\Gamma^{\mu_1}_{\kappa\nu_2}g^{\mu_2\kappa}W^{\mu_3\mu_4}_{(E)\nu_3\nu_4} \,,\end{aligned} \tag{4.22}$$

when the EOM hold.

In section 3.2.2 we showed (4.14) that, when I_{MM} is written in Gauss-normal coordinates, can be put in the form

$$\begin{aligned}\delta I_{MM} = \int_{\partial M} d^3x\sqrt{-h}\delta^{[i_1i_2i_3]}_{[j_1j_2j_3]}\Big\{&\Big[2\delta K^{j_1}_{i_1} + K^{m}_{i_1}\left(h^{-1}\delta h\right)^{j_1}_{m}\Big]W^{j_2j_3}_{(E)i_2i_3} \\ &+ 2ND^{j_2}W^{j_3\rho}_{(E)i_2i_3}\left(h^{-1}\delta h\right)^{j_1}_{i_1}\Big\} \,.\end{aligned} \tag{4.23}$$

The presence of the variation of the extrinsic curvature δK makes it not possible to have a well-defined Dirichlet problem for the metric h_{ij}, making the definition of the quasilocal tensor a cumbersome problem to solve.

However, considering the FG expansion of the metric, we find the asymptotic behavior of the following quantities,

$$\begin{aligned} \sqrt{-h} &= \frac{\sqrt{g_{(0)}}}{\rho^{3/2}} + \mathcal{O}\left(\rho^{-1/2}\right) , && (4.24) \\ \left(h^{-1}\delta h\right)^j_\ell &= \left(g^{-1}_{(0)}\delta g_{(0)}\right)^j_\ell + \mathcal{O}(\rho) , && (4.25) \\ K^i_j(h) &= \frac{1}{\ell}\delta^i_j - \ell\rho S^i_j\left(g_{(0)}\right) + \mathcal{O}(\rho^{3/2}) , && (4.26) \end{aligned}$$

whereas for the fall-off of the components of the Weyl tensor we get

$$\begin{aligned} W^{i\rho}_{jk} &= \mathcal{O}\left(\rho^2\right) , && (4.27) \\ W^{ik}_{jm} &= \mathcal{O}\left(\rho\right) . && (4.28) \end{aligned}$$

Summing up all the above quantities, we realize that the first and the third terms in (4.23) are subdominant, as their fall-off is of order $\mathcal{O}\left(\rho\right)$, and they do not contribute at the conformal boundary. Thus, the only finite contribution is coming from the second term, which provides a well-defined variational problem for the conformal boundary metric $g_{(0)ij}$. This feature makes the topological regularization compatible with holography, as $g_{(0)ij}$ corresponds to the background metric of the boundary CFT.

Both the renormalized action for Einstein AdS gravity (4.21) and its corresponding variation (4.22) will play a crucial role in the analysis below, where we will show the equivalence of CG and EH gravity.

4.4 From Conformal to Einstein Gravity

EW gravity is described by the action

$$I^{(EW)} = \frac{1}{16\pi G}\int_M d^4x\sqrt{-g}\left[(R-2\Lambda) + \gamma W^{\alpha\beta}_{\mu\nu}W^{\mu\nu}_{\alpha\beta}\right] , \tag{4.29}$$

where γ is a constant proportional to the CG coupling which reads $\gamma = 16\pi G\alpha_{CG}$. By construction, the theory consists of two distinct parts, which are reflected in the EOM

$$G^\nu_\mu - \gamma B^\nu_\mu = 0 , \tag{4.30}$$

where

$$G^{\mu}_{\nu} = R^{\nu}_{\mu} - \frac{1}{2}R\delta^{\nu}_{\mu} - \frac{3}{\ell^2}\delta^{\nu}_{\mu} = -\frac{1}{4}\delta^{[\nu\alpha\beta]}_{[\mu\gamma\delta]}\left(R^{\gamma\delta}_{\alpha\beta} + \frac{1}{\ell^2}\delta^{[\gamma\delta]}_{[\alpha\beta]}\right), \tag{4.31}$$

is the Einstein tensor.

For the discussion below, it is useful to define what we call Einstein spacetimes. The direct response would be that Einstein spaces are the ones defining a vanishing Einstein tensor. But there is an even more generic definition that involves the Ricci tensor itself. As Einstein are called spaces whose traceless Ricci tensor vanishes, i.e.,

$$R^{T}_{\mu\nu} = R_{\mu\nu} - \frac{1}{D}Rg_{\mu\nu} = 0\,, \tag{4.32}$$

where D corresponds to the spacetime dimension. Indeed, in $D = 4$ EH gravity, $R^{T}_{\mu\nu} = 0$, leading to the following relation for the Ricci tensor: $R_{\mu\nu} = -\frac{3}{\ell^2}g_{\mu\nu}$.

Taking the trace of the EW EOM in Eq.(4.30), the Ricci tensor becomes

$$R_{\mu\nu} = -\frac{3}{\ell^2}g_{\mu\nu} + \gamma B_{\mu\nu}\,. \tag{4.33}$$

This relation indicates that the class of solutions of the theory consists of a broader set of spacetimes than the Einstein ones. The deviation from the Einstein branch is expressed from the term linear in the Bach tensor. Thus, the presence of quadratic-curvature terms in the action enlarges the solution space.

We have to stress out that arbitrary quadratic couplings in the curvature would lead to a possibility to produce multiple AdS vacua. It is the specific value of the couplings which correspond to the Weyl squared term that do not modify the asymptotic behavior and maintain a unique AdS vacuum.

Based on the previous discussion for EW gravity, and considering that Einstein spacetimes is a subset of the CG solutions, we assume that the most generic solution to CG is a deviation from the Einstein branch. Thus, we conjecture that the higher-derivative contributions in the CG action are captured by the Bach tensor in Eq.(4.33). The separation at the level of the curvature is translated into a decomposition of the Weyl tensor into an Einstein and a non-Einstein part of the form $W = W_{(E)} + W_{(NE)}$.

More specifically, substituting the Ricci decomposition (4.33) into the Schouten tensor (2.19), one gets

$$S^{\alpha}_{\mu} = -\frac{1}{2}\left(\frac{1}{\ell^2}\delta^{\alpha}_{\mu} - \gamma B^{\alpha}_{\mu}\right). \tag{4.34}$$

Plugging this expression into the Weyl tensor (2.18), we identify the Einstein part as the one given by the Eq.(4.20), while the NE part is described as the skew-symmetric product of the Bach tensor and the metric

$$W^{\alpha\beta}_{(NE)\mu\nu} = -\frac{\gamma}{2}(B^{\alpha}_{\mu}\delta^{\beta}_{\nu} - B^{\beta}_{\mu}\delta^{\alpha}_{\nu} - B^{\alpha}_{\nu}\delta^{\beta}_{\mu} + B^{\beta}_{\nu}\delta^{\alpha}_{\mu})\,. \tag{4.35}$$

Hence, using the Weyl decomposition in the CG action (4.1), it adopts the form

$$I_{CG} = \frac{\alpha_{CG}}{4} \int_M d^4x \sqrt{-g} \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} \Big(W^{\mu_1\mu_2}_{(E)\nu_1\nu_2} W^{\mu_3\mu_4}_{(E)\nu_3\nu_4} + 2W^{\mu_1\mu_2}_{(E)\nu_1\nu_2} W^{\mu_3\mu_4}_{(NE)\nu_3\nu_4} + W^{\mu_1\mu_2}_{(NE)\nu_1\nu_2} W^{\mu_3\mu_4}_{(NE)\nu_3\nu_4} \Big) . \tag{4.36}$$

Would the coupling be chosen as $\alpha_{CG} = \ell^2/64\pi G$, the first term would be identified as the MacDowell-Mansouri form of the renormalized Einstein-AdS action. Expanding different parts of the decomposition, the CG action equivalently can be written as

$$I_{CG} = I_{MM} - \frac{\ell^2}{16\pi G} \gamma \int_M d^4x \sqrt{-g} \delta^{[\nu_1\nu_2]}_{[\mu_1\mu_2]} (G^{\mu_1}_{\nu_1} - \frac{\gamma}{2} B^{\mu_1}_{\nu_1}) B^{\mu_2}_{\nu_2} , \tag{4.37}$$

where Eq.(4.21) was taken into account and $\gamma = \ell^2/4$.

It is evident from this formulation that, for Einstein spacetimes, namely vanishing Bach spacetimes, only the contribution from I_{MM} survives, expressing the equivalence between Einstein-AdS and CG. At this point, it is useful to comment that the vanishing of the Bach tensor extends the validity of the equivalence between the two actions to more generic spacetimes than the Einstein ones, i.e., Bach-flat spaces. Further restrictions are imposed when we pass at the level of the variation of the action.

Indeed, when the EOM for CG hold and the decomposition of the Weyl tensor is included, the surface term in Eq.(4.9) becomes

$$\delta I_{CG} = \alpha_{CG} \int_{\partial M} d^3x \sqrt{-h} \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} \Big[n_{\nu_1} \delta\Gamma^{\mu_1}_{\lambda\nu_2} g^{\mu_2\lambda} \Big(W^{\mu_3\mu_4}_{(E)\nu_3\nu_4} + W^{\mu_3\mu_4}_{(NE)\nu_3\nu_4} \Big) + n^{\mu_1} \nabla_{\nu_1} W^{\mu_2\mu_3}_{\nu_2\nu_3} \Big(g^{-1}\delta g \Big)^{\mu_4}_{\nu_4} \Big] , \tag{4.38}$$

where we used the Bianchi identity in order to eliminate the $W_{(E)}$ part in the covariant derivative. Inserting the explicit form of the non-Einstein part of the Weyl tensor (4.35), the surface term becomes

$$\delta I_{CG} = \alpha_{CG} \int_{\partial M} d^3x \sqrt{-h} \delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]} n_{\nu_1} \delta\Gamma^{\mu_1}_{\lambda\nu_2} g^{\mu_2\lambda} W^{\mu_3\mu_4}_{(E)\nu_3\nu_4} - 2\alpha_{CG}\gamma \int_{\partial M} d^3x \sqrt{-h} \delta^{[\nu_1\nu_2\nu_3]}_{[\mu_1\mu_2\mu_3]} \Big[n_{\nu_1} \delta\Gamma^{\mu_1}_{\lambda\nu_2} g^{\mu_2\lambda} B^{\mu_3}_{\nu_3} + n^{\mu_1} \nabla_{\nu_1} B^{\mu_2}_{\nu_2} \Big(g^{-1}\delta g \Big)^{\mu_3}_{\nu_3} \Big] . \tag{4.39}$$

For the same value of the CG coupling constant as before,I.e., $\alpha_{CG} = \ell^2/64\pi G$, the first term can be identified as the variation of the renormalized AdS action in Eq.(4.22). Thus, the surface term that results from the CG action (4.39), adopts the form

$$\delta I_{CG} = \delta I_{MM} - \frac{\ell^4}{128\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\delta^{[\nu_1\nu_2\nu_3]}_{[\mu_1\mu_2\mu_3]}\left[n_{\nu_1}\delta\Gamma^{\mu_1}_{\kappa\nu_2}g^{\mu_2\kappa}B^{\mu_3}_{\nu_3}+ \right.$$
$$\left. +n^{\mu_1}\nabla_{\nu_1}B^{\mu_2}_{\nu_2}\left(g^{-1}\delta g\right)^{\mu_3}_{\nu_3}\right]. \tag{4.40}$$

It has to be noted that, in general, $\delta\Gamma$ in the first term of (4.39) contains non-Einstein contributions from the metric, such that it cannot be matched to the Eq.(4.22) without imposing further restrictions on the boundary conditions of the theory. In the holographic context, these contributions are recognized as new sources on the conformal boundary and arise as additional terms in the FG expansion.

More specifically, in the FG gauge, an ALAdS spacetime is foliated as

$$ds^2 = \frac{\ell^2}{z^2}dz^2 + \frac{1}{z^2}g_{ij}(z,x)dx^i dx^j\,, \tag{4.41}$$

with the power series of the metric near the conformal boundary $z = 0$ written as

$$g_{ij}(z,x) = g_{(0)ij} + zg_{(1)ij} + z^2 g_{(2)ij} + z^3 g_{(3)ij} + \dots\,. \tag{4.42}$$

The term linear in z would be absent in case of Einstein gravity, because the odd powers of z are eliminated by the EOM. In CG this is no longer the case and the presence of the term $zg_{(1)}$ represents the non-Einstein branch of the theory. Maldacena's argument [31], consists on switching off the non-Einstein modes by imposing proper Neumann boundary conditions, i.e.,$\partial_z g_{ij}(z,x) = 0$, recovering the Einstein branch of solutions.

Instead, in our formulation, constraining in the Einstein subset of CG solutions implies both a vanishing Bach tensor and proper Neumann boundary conditions that removes the non-Einstein modes. Based on this, our derivation provides an explicit proof of the equivalence between Einstein gravity with a cosmological constant and CG, both at the level of the action and the variation of it.

Chapter 5

Critical Gravity

5.1 Introduction

CrG is the second main paradigm of a higher-curvature theory of gravity in four dimensions, after CG, which was studied extensively in the last chapter. As it was clearly stated, the main advantage of these models is that they manage to solve the problem of renormalizability [20], [25], that arises in GR [19], providing a finite UV behavior and a possibly consistent theory of quantum gravity. Thus, these toy models reveal many interesting aspects of a quantum theory of gravity.

More specifically, CrG is a special case of a class of theories characterized by the presence of quadratic curvature terms added to the EH action. The most general form of this class of theories in 4D is given by

$$I = \frac{1}{16\pi G}\int_{M} d^4x\sqrt{-g}\left(R - 2\Lambda + \alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2\right) , \tag{5.1}$$

where α and β are arbitrary couplings and $\Lambda = -3/\ell^2$ is the cosmological constant. No Riemann squared term appears in the above expression, as it can always be expresses in terms of the GB topological invariant and the above quadratic curvature terms. The GB in 4D is a topological invariant, namely it doesn't affect the EOM, but it modifies the boundary dynamics.

The presence of quadratic-curvature terms lead to four-derivative EOM, that correspond to a spectrum that consists of massless spin-2, massive spin-2 and massive scalar modes [20], [38], [39]. The massive graviton is actually a ghost mode (it has negative energy), as known from a generic higher-derivative theory [38], rendering the theory non-unitary.

The problem of the ghost mode can be circumvented by flipping the conventional sign of the EH term, but in this case the mass of the Schwarzschild-AdS turns negative. The inconsistency coming from the opposite signs of the black hole mass and the energy of the perturbations makes the model unphysical. Similar behavior has been reported in the case of 3D massive gravity theories (NMG,TMG) [22], [23]. They possess features that can be extended to higher dimensions and overcome the pathologies of the theory.

The property that is of major interest for the construction of a physically consistent theory, is criticality. This phenomenon arises in specific points in the parametric space of the coupling constants where the linearized EOM are degenerate. At these specific points, the scalar modes vanish while the massive gravitons turn to massless. Moreover, both the energy of the propagating excitations and the black hole mass are zero, leading to a model which is unitary and without ghosts. The concept of criticality when extended in 4D and applied in the quadratic-curvature gravity action (5.1) gives rise to Critical Gravity [24].

New modes arise at the critical point, which have a logarithmic dependence in the radial coordinate near the boundary. In general, imposing the standard AdS boundary conditions, the logarithmic modes can be discarded from the spectrum of the theory as they have a faster fall-off. The resulting theory is somewhat trivial, in the sense that both the massless excitations and the Schwarzschild-AdS black hole have zero energy [10], [24]. The triviality of the theory is justified by the on-shell equivalence between Einstein-AdS and CG when the non-Einstein modes are absent [9], [31]. We discuss this issue below.

The structure of CrG gets richer when a relaxed set of AdS boundary conditions is taken into account [40]–[42], as the presence of the logarithmic modes modify not only the spectrum but also the asymptotic structure of the theory. The spacetime is no longer asymptotically AdS and a standard holographic description breaks down. This is evident from the presence of logarithmic terms in the relaxed FG expansion.

The modification of the asymptotics is expressed as a function of the coefficient of the leading order logarithmic term,i.e, $b_{(0)ij}$. In order to maintain a valid holographic description in the presence of logarithmic modes, the asymptotic region has to be modified minimally from the ALAdS fall-off. In order to circumvent the problem, we assume that $b_{(0)ij}$ is small. The presence of new modes switches on a new source in the boundary. Indeed, the leading order logarithmic coefficient plays the role of the source of an operator living on the boundary CFT, which is a LCFT instead of a CFT.

LCFTs are characterised by the logarithmic terms that arise in the operator product expansion (OPE), preserving the conformal invariance [43]. The logarithmic operators correspond to a generalization of the primary operators for non-diagonalizable matrices and they appear as conjugate to zero norm primary states with degenerate scaling dimensions [44]. As a consequence, the Hamiltonian of these states in non-Hermitian, describing systems where unitarity fails.

CrG gives us insight and the appropriate tools to study further the properties of LCFTs, in a holographic context. In this chapter, we take advantage of the equivalence between Einstein-AdS gravity and CG and derive a new on-shell formula for CrG. Based on this new formulation we propose a new set of counterterms that cancels the divergences coming from the non-Einstein modes, simplifying the derivation of the holographic correlation functions.

5.2 Critical Gravity action

CrG is defined in a unique point in the parametric space of a generic quadratic-curvature action. The corresponding coupling constants lead to degenerate EOM, in a way that all the massive excitations of the theory vanish. Thus, the identification of the critical coupling constants in (5.1) requires to write the EOM in the linearized form around a background, in order to determine the dynamics of the propagating excitations. The linearization procedure [24] requires the presence of a perturbation $h_{\mu\nu}$, propagating in the background $\tilde{g}_{\mu\nu}$, such that the metric of the spacetime is given by $g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$.

The field equations at first order in the perturbation $h_{\mu\nu}$ are

$$\begin{aligned}\delta\left(G_{\mu\nu} + E_{\mu\nu}\right) &= \left[1 + 2\Lambda\left(\alpha + 4\beta\right)\right] G^{L}_{\mu\nu} + \alpha\left[\left(\bar{\Box} - \frac{2\Lambda}{3}\right) G^{L}_{\mu\nu} - \frac{2\Lambda}{3} R^{L} \bar{g}_{\mu\nu}\right] + \\ &+ \left(\alpha + 2\beta\right)\left[-\bar{\nabla}_{\mu}\bar{\nabla}_{\nu} + \bar{g}_{\mu\nu}\bar{\Box} + \Lambda\bar{g}_{\mu\nu}\right] R^{L} = 0\,, \end{aligned} \tag{5.2}$$

where $G^{L}_{\mu\nu}$ and R^{L} are the linearized versions of the Einstein tensor and the Ricci scalar, given by

$$G^{L}_{\mu\nu} = R^{L}_{\mu\nu} - \frac{1}{2} R^{L} \bar{g}_{\mu\nu} - \Lambda h_{\mu\nu}\,, \tag{5.3}$$

$$R^{L}_{\mu\nu} = \bar{\nabla}^{\mu}\bar{\nabla}_{(\nu} h_{\nu)\lambda} - \frac{1}{2}\bar{\Box} h_{\mu\nu} - \frac{1}{2}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu} h\,, \tag{5.4}$$

$$R^{L} = \bar{\nabla}^{\mu}\bar{\nabla}^{\nu} h_{\mu\nu} - \bar{\Box} h - \Lambda h\,. \tag{5.5}$$

Here $E_{\mu\nu}$ quantifies all the four-derivative contributions of the EOM. Taking the trace and considering the gauge condition $\bar{\nabla} h_{\mu\nu} = \bar{\nabla}_{\nu} h$, we obtain

$$g^{\mu\nu}\delta\left(G_{\mu\nu} + E_{\mu\nu}\right) = 0 = \Lambda\left[h - 2\left(\alpha + 3\beta\right)\bar{\Box} h\right]\,, \tag{5.6}$$

which corresponds to the EOM of the propagating scalar mode. We note that the choice $\alpha = -3\beta$ eliminates the massive scalar mode and is equivalent to the imposition of the traceless condition $h = 0$. Hence, the field equations for the excitations, in the absence of massive scalar modes, acquire the form

$$\left(\bar{\Box} - \frac{2\Lambda}{3}\right)\left(\bar{\Box} - \frac{2\Lambda}{3} - \frac{2\Lambda\beta + 1}{3\beta}\right) h_{\mu\nu} = 0\,. \tag{5.7}$$

These EOM provide the spectrum of the theory, which consists of a massless graviton in an AdS background, represented by the first parentheses, and a massive spin-2 field, if $2\Lambda\beta + 1 \neq 0$, represented by the second. Notice that, for $\beta = -1/2\Lambda$ the massive graviton turns massless and the EOM (5.7) are degenerate.

Hence, the critical coupling constants correspond to the values $\alpha = -3\beta$ and $\beta = -1/2\Lambda$. Indeed, for this value the EOM acquire the form

$$\left(\bar{\Box} - \frac{2\Lambda}{3}\right)^2 h_{\mu\nu} = 0\,, \tag{5.8}$$

representing only massless and logarithmic modes. The second class

With the critical couplings at hand, the CrG action in $4D$ becomes

$$I_{critical} = \frac{1}{16\pi G}\int\limits_{M} d^4x\sqrt{-g}\left[R - 2\Lambda + \frac{3}{2\Lambda}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)\right]\,, \tag{5.9}$$

where $\Lambda = -3/\ell^2$ is the cosmological constant.

Alternatively, considering that the specific quadratic-curvature combination corresponds to the difference between the Weyl squared and GB term $\mathcal{E}_4$, the CrG action can be equivalently written off-shell as

$$I_{critical} = \frac{1}{16\pi G}\int\limits_{M} d^4x\sqrt{-g}\left[\left(R + \frac{6}{\ell^2}\right) + \frac{\ell^2}{4}\left(\mathcal{E}_4 - W^{\mu\nu\alpha\beta}W_{\alpha\beta\mu\nu}\right)\right]\,. \tag{5.10}$$

The GB term does not modify the bulk EOM but it does affect the boundary dynamics. Here, the coupling constant of the GB term is the appropriate one to generate the MacDowell-Mansouri action I_{MM}, when summed up to the EH action, as discussed in Chapter 4. Thus, the CrG action is split in two parts,

$$I_{critical} = I_{MM} - \frac{\ell^2}{64\pi G}\int\limits_{M} d^4x\sqrt{-g}W^{\alpha\beta\mu\nu}W_{\alpha\beta\mu\nu}\,, \tag{5.11}$$

previously seen in [9], [45]. In what follows, we will refer to the Weyl squared term of the action in (5.11) as CG, even though the coupling is fixed,

$$I_{CG} = \frac{\ell^2}{256\pi G}\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}W^{\mu_1\mu_2}_{\nu_1\nu_2}W^{\mu_3\mu_4}_{\nu_3\nu_4}\,. \tag{5.12}$$

Note that the coupling in front of the CG action has the correct value to realize the equivalence between CG and the Einstein-AdS gravity when evaluated for Einstein spacetimes, as discussed in the last chapter.

Eqs.(5.11) and (5.12) make manifest the vanishing of $I_{critical}$ for Einstein spacetimes, where $W^{\mu\nu}_{\alpha\beta} = 0$, as it was shown in [9], [45], explaining the triviality of the theory for this branch of solutions.

5.3 Field equations and the on-shell action

The CrG action has three contributions, the CG term (5.12), the EH action and the GB topological invariant that does not modify the EOM. It is a special case of the EW gravity, seen in the last chapter, with fixed coupling constant $\gamma = -\ell^2/4$ and the field equations given by

$$G^{\mu}_{\nu} + \frac{\ell^2}{4} B^{\mu}_{\nu} = 0 \,, \tag{5.13}$$

where G^{μ}_{ν} is the Einstein tensor with negative cosmological constant (4.31) and B^{μ}_{ν} is the Bach tensor (4.10).

Taking the trace of (5.13), the contribution of the Bach is zero and the Ricci scalar recovers its value from General Relativity ($R = -12/\ell^2$) and there is a unique AdS vacuum in the theory. The absence of multiple AdS vacua shows that the Weyl squared term corresponds to the unique quadratic-curvature combination that does not produce the degeneracy of the vacuum.

Substituting the Ricci scalar into the EOM, we get

$$R_{\mu\nu} = -\frac{3}{\ell^2} g_{\mu\nu} - \frac{\ell^2}{4} B_{\mu\nu} \,. \tag{5.14}$$

As previously mentioned, the above relation expresses the deviation of a general solution of CrG from Einstein spaces. The term that introduces the NE branch of the theory is linear in the Bach tensor.

The curvature decomposition Eq.(5.14) leads to a splitting of the Weyl tensor in two parts: i) Einstein part $W^{\alpha\beta}_{(E)\mu\nu}$ (4.20) and ii) non-Einstein piece $W^{\alpha\beta}_{(NE)\mu\nu}$, which reads

$$W^{\alpha\beta}_{(NE)\mu\nu} = \frac{\ell^2}{8} \left(B^{\alpha}_{\mu} \delta^{\beta}_{\nu} - B^{\beta}_{\mu} \delta^{\alpha}_{\nu} - B^{\alpha}_{\nu} \delta^{\beta}_{\mu} + B^{\beta}_{\nu} \delta^{\alpha}_{\mu} \right) \,, \tag{5.15}$$

after plugging in the value of γ into Eq.(4.35). Thus, the total Weyl tensor can be written as

$$W^{\alpha\beta}_{\mu\nu} = W^{\alpha\beta}_{(E)\mu\nu} + W^{\alpha\beta}_{(NE)\mu\nu} \,. \tag{5.16}$$

As we told before, the coupling constant of the CG term has the right value that allows us to rewrite the action as a function of the renormalized Einstein-AdS action and higher-derivative contributions, given by Eq.(4.37). As a consequence, the first term of the CG action cancels the $I^{(E)}_{ren}$ part of CrG (5.11) and the remaining piece is

$$I_{critical} = -\frac{\ell^4}{64\pi G} \int\limits_M d^4x \sqrt{-g} \delta^{[\kappa\lambda]}_{[\mu\nu]} \left(\frac{\ell^2}{8} B^{\mu}_{\kappa} + G^{\mu}_{\kappa} \right) B^{\nu}_{\lambda} \,. \tag{5.17}$$

Taking into consideration that the EOM (5.13) hold, the CrG action can be cast into the form

$$\begin{aligned} I_{critical} &= \frac{\ell^6}{512\pi G}\int_M d^4x\sqrt{-g}\delta^{[\kappa\lambda]}_{[\mu\nu]}B^\mu_\kappa B^\nu_\lambda \\ &= -\frac{\ell^6}{512\pi G}\int_M d^4x\sqrt{-g}B^\mu_\kappa B^\kappa_\mu\,. \end{aligned} \tag{5.18}$$

Hence, we conclude that the Einstein spaces have vanishing contribution in the on-shell CrG action. The only non-vanishing contribution is coming from the non-Einstein spaces with a term that is quadratic in the Bach tensor. Due to the fact that Einstein spaces are Bach flat, the $I_{critical}$ is zero. As a consequence both the mass and the entropy of the Schwarzschild-AdS black hole is 0, making this sector trivial.

5.4 Noether-Wald charges in Critical Gravity

A different path for the calculation of energy in CrG, based on non-perturbative arguments, is presented below. Unlike the Deser-Tekin [46], [47] prescription, based on the analysis in linearized limit, we define the conserved charges of the theory using the Noether-Wald method [48], [49].

Starting from a Lagrangian of the form $\mathcal{L}\left(g_{\mu\nu}, R_{\mu\nu\alpha\beta}\right)$ whose dynamics describes geometries such that their isometries given by a set of Killing vectors $\{\xi^\mu\}$, the Noether current associated to diffeomorphisms $\delta x^\mu = \xi^\mu$ reads

$$\sqrt{-g}J^\mu = \Theta^\mu\left(\delta_\xi g\right) + \Theta^\mu\left(\delta_\xi \Gamma\right) + \sqrt{-g}\mathcal{L}\xi^\mu\,. \tag{5.19}$$

Here, Θ^μ is the surface term resulting from the on-shell variation of the Lagrangian. The fact that the manifold, endowed with a metric $g_{\mu\nu}$, possesses isometries is expressed by the Killing equation, $\delta_\xi g_\mu = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0$. In consequence, the first term in (5.19) vanishes and the Noether current is written down as

$$J^\mu = 2E^{\mu\nu}_{\alpha\beta}\left(\nabla_\nu\nabla^\alpha\xi^\beta + R^{\alpha\beta}_{\nu\sigma}\xi^\sigma\right) + \mathcal{L}\xi^\mu\,, \tag{5.20}$$

where $E^{\mu\nu}_{\alpha\beta}$ is the functional derivative of the Lagrangian with respect to the Riemann tensor, i.e.,

$$E^{\mu\nu}_{\alpha\beta} = \frac{\delta\mathcal{L}}{\delta R^{\alpha\beta}_{\mu\nu}}\,. \tag{5.21}$$

Note that $E^{\mu\nu}_{\alpha\beta}$ has the same symmetries as the Riemann tensor. When evaluated on-shell, the last two terms of Eq.(5.20) vanish, because they construct the EOM contracted by the Killing field, and the surviving part can be equivalently be written as

$$J^{\mu} = 2\nabla_{\nu}\left(E^{\mu\nu}_{\alpha\beta}\nabla^{\alpha}\xi^{\beta}\right) , \tag{5.22}$$

where the Bianchi identity was taken into account. The conserved current is now a total derivative, what allows to write the conserved charge as a codimension-2 surface integral. In this case, the conserved charge of the theory is given by

$$Q^{\mu}\left[\xi\right] = 2\int\limits_{\Sigma} dS_{\nu}E^{\mu\nu}_{\alpha\beta}\nabla^{\alpha}\xi^{\beta} , \tag{5.23}$$

where Σ is the codimension-2 surface corresponding to $r = const$ and $t = const$.

Applying the Noether construction in CrG requires the computation of the tensor $E^{\mu\nu}_{\alpha\beta}$. The derivation is much simplified when using the alternative form of the action (5.11), containing two separate parts, a MacDowell-Mansouri term [50] and a Weyl-squared term, respectively. Each term contributes separately to the functional derivative of the Riemann tensor as

$$\begin{aligned} E^{\mu\nu}_{\alpha\beta} &= \left(E^{\mu\nu}_{\alpha\beta}\right)^{MM} + \left(E^{\mu\nu}_{\alpha\beta}\right)^{CG} \\ &= \frac{\ell^2}{128\pi G}\delta^{[\mu\nu\sigma\lambda]}_{[\alpha\beta\gamma\delta]}\left[\left(R^{\gamma\delta}_{\sigma\lambda} + \frac{1}{\ell^2}\delta^{[\gamma\delta]}_{[\sigma\lambda]}\right) - W^{\gamma\delta}_{\sigma\lambda}\right] . \end{aligned} \tag{5.24}$$

Plugging this result into the Noether-Wald formula (5.23) gives

$$Q^{\mu}\left[\xi\right] = \frac{\ell^2}{64\pi G}\int\limits_{\Sigma} dS_{\nu}\delta^{[\mu\nu\sigma\lambda]}_{[\alpha\beta\gamma\delta]}\nabla^{\alpha}\xi^{\beta}\left[\left(R^{\gamma\delta}_{\sigma\lambda} + \frac{1}{\ell^2}\delta^{[\gamma\delta]}_{[\sigma\lambda]}\right) - W^{\gamma\delta}_{\sigma\lambda}\right] . \tag{5.25}$$

Due to the fact that Weyl tensor for Einstein spaces, given by Eq.(4.20), is equivalent to the AdS curvature, namely the parentheses in (5.25), the conserved charges in CrG are identically zero.

Hence, we showed the triviality of CrG when we are constrained in the Einstein branch of the theory. This result, previously seen in [10], confirms that non-trivial contribution to the charges is coming from the non-Einstein modes of the theory, as we have shown in the last section and in Ref.[9].

5.5 Surface terms

In what follows, we are computing the surface terms arising from the variation of the CrG action with a goal to be used in the derivation of the derivation of the holographic correlation functions, considering the expressions already derived for the CG case. We have shown that the renormalized Einstein-AdS action, defined by plugging the GB term on top of the EH action, is written as the MacDowell-Mansouri action for the AdS group (4.21) while the variation of the action is given by (4.22). Furthermore, for the exact value of the coupling constant in front of the Weyl^2 term in the CrG action, we have shown that the variation of the CG action is given by Eq.(4.40). Once more, the contributions coming from the renormalized Einstein action are canceled giving rise only to the terms that depend on the Bach tensor, as it is shown below,

$$\begin{aligned} \delta I_{critical} &= \delta I_{ren}^{(E)} - \delta I_{CG} \\ &= -\frac{\ell^4}{128\pi G}\int\limits_{\partial M} d^3x\sqrt{-h}\delta^{[\nu_1\nu_2\nu_3]}_{[\mu_1\mu_2\mu_3]}\left[n_{\nu_1}\delta\Gamma^{\mu_1}_{\kappa\nu_2}g^{\mu_2\kappa}B^{\mu_3}_{\nu_3} + n^{\mu_1}\nabla_{\nu_1}B^{\mu_2}_{\nu_2}\left(g^{-1}\delta g\right)^{\mu_3}_{\nu_3}\right]. \end{aligned} \tag{5.26}$$

We realize that the variation of the CrG action is linear in the Bach tensor and vanishes for Einstein spacetimes. This fact actually explains the vanishing energy for Einstein spacetimes, including the energy of the massless excitations, previously seen in Refs.[24], [39], [51].

Furthermore, using the EOM, the action variation (5.26) can be put in the equivalent form

$$\begin{aligned} \delta I_{critical} = \frac{\ell^2}{32\pi G}\int\limits_{\partial M} d^3x\sqrt{-h}\delta^{[\nu_1\nu_2\nu_3]}_{[\mu_1\mu_2\mu_3]}\Big[n_{\nu_1}\delta\Gamma^{\mu_1}_{\kappa\nu_2}g^{\kappa\mu_2}G^{\mu_3}_{\nu_3} + \\ + n^{\mu_1}\nabla_{\nu_1}G^{\mu_2}_{\nu_2}\left(g^{-1}\delta g\right)^{\mu_3}_{\nu_3}\Big]. \end{aligned} \tag{5.27}$$

The variation of the CrG action plays crucial role in the holographic description, as it is the quantity that determines the dual stress-energy tensor. What the form (5.27) shows us, is that all the non-vanishing contributions are coming from the non-Einstein spacetimes, namely solutions with a non-vanishing Einstein tensor. A more general proof that Einstein spacetimes have zero energy can be made by using Noether-Wald charges [10].

The next step towards the derivation of the holographic correlation functions is to express the surface term (5.27) in the radial foliation with the help of Gaussian coordinates

$$ds^2 = N^2(\rho)\,d\rho^2 + h_{ij}(\rho,x)\,dx^i dx^j\,. \tag{5.28}$$

Starting from the first term of Eq.(5.27), we get

$$\delta^{[\nu_1\nu_2\nu_3]}_{[\mu_1\mu_2\mu_3]} n_{\nu_1} \delta\Gamma^{\mu_1}_{\kappa\nu_2} g^{\kappa\mu_2} G^{\mu_3}_{\nu_3} = N\delta^{[ij]}_{[k\ell]} \left(\delta\Gamma^{\rho}_{mi} h^{mk} G^{\ell}_{j} - \delta\Gamma^{k}_{\rho i} g^{\rho\rho} G^{\ell}_{j} + \delta\Gamma^{k}_{mi} h^{m\ell} G^{\rho}_{j} \right) , \quad (5.29)$$

where the first two terms become

$$N\delta^{[ij]}_{[k\ell]} \left[\delta \left(\frac{1}{N} K_{mi} \right) h^{km} G^{\ell}_{j} - \delta\Gamma^{k}_{\rho i} g^{\rho\rho} G^{\ell}_{j} \right] = \delta^{[ij]}_{[k\ell]} [K^{m}_{i} \left(h^{-1}\delta h \right)^{k}_{m} + 2\delta K^{k}_{i}] G^{\ell}_{j} , \quad (5.30)$$

and the last one is

$$\int\limits_{\partial M} d^3x \sqrt{-h} N \delta^{[ij]}_{[k\ell]} \delta\Gamma^{k}_{mi} h^{\ell m} G^{\rho}_{j} = - \int\limits_{\partial M} d^3x \sqrt{-h} N \delta^{[ij]}_{[k\ell]} \left(h^{-1}\delta h \right)^{k}_{i} D^{\ell} G^{\rho}_{j} . \quad (5.31)$$

Taking into account these contributions, the first term of the variation of the action is written in the radial foliation as

$$\begin{aligned} \delta^{[\nu_1\nu_2\nu_3]}_{[\mu_1\mu_2\mu_3]} n_{\nu_1} \delta\Gamma^{\mu_1}_{\kappa\nu_2} g^{\kappa\mu_2} G^{\mu_3}_{\nu_3} = \delta^{[ij]}_{[k\ell]} \Bigg[\left(K^{m}_{i} \left(h^{-1}\delta h \right)^{k}_{m} + 2\delta K^{k}_{i} \right) G^{\ell}_{j} - \\ - N D^{k} G^{\rho}_{i} \left(h^{-1}\delta h \right)^{\ell}_{j} \Bigg] . \end{aligned} \quad (5.32)$$

For the second term in the variation (5.27), we get

$$\delta^{[\nu_1\nu_2\nu_3]}_{[\mu_1\mu_2\mu_3]} n^{\mu_1} \nabla_{\nu_1} G^{\mu_2}_{\nu_2} \left(g^{-1}\delta g \right)^{\mu_3}_{\nu_3} = \delta^{[ij]}_{[k\ell]} \frac{1}{N} \left(\nabla_{\rho} G^{k}_{i} - \nabla_{i} G^{k}_{\rho} \right) \left(h^{-1}\delta h \right)^{\ell}_{j} . \quad (5.33)$$

Hence, the variation of the CrG action in Gauss-normal coordinates becomes

$$\begin{aligned} \delta I_{critical} &= \frac{\ell^2}{32\pi G} \int\limits_{\partial M} d^3x \sqrt{-h} \delta^{[ij]}_{[k\ell]} \Bigg[\left(2\delta K^{k}_{i} + K^{m}_{i} \left(h^{-1}\delta h \right)^{k}_{m} \right) G^{\ell}_{j} \\ &+ \frac{1}{N} \left(\nabla_{\rho} G^{k}_{i} - \nabla_{i} G^{k}_{\rho} - N^2 D^{k} G^{\rho}_{i} \right) \left(h^{-1}\delta h \right)^{\ell}_{j} \Bigg] . \end{aligned} \quad (5.34)$$

Note that the variation of the CrG consists of terms which are linear in the Einstein tensor. It is the asymptotic behavior of the various components of the Einstein tensor that will determine the finiteness of the holographic correlation functions, which are determined from the functional derivative of (5.34) with respect to the sources.

5.6 Holographic Renormalization in Critical Gravity

Due to the degeneracy of the linearized EOM in CrG, the spectrum of the theory contains new modes which have logarithmic dependence in the radial coordinate. The logarithmic modes have a slower fall-off than the standard AAdS spacetimes and choosing suitable boundary conditions they can be discarded. In this case, the AdS/CFT dictionary is still valid, because the Einstein modes become the only surviving modes in the spectrum, and they become the sources at the boundary coupled to CFT operators.

By relaxing the boundary conditions, one switches-on new independent sources associated with the propagating logarithmic modes. The new sources correspond to a LCFT, instead of the CFT, providing an intuition on aspects of the AdS/LCFT, such as the computation of the holographic correlation functions.

In Ref.[40], [52] it was proposed a relaxed set of AdS boundary conditions which is consistent with the logarithmic branch of the theory. For the radial foliation (5.28), with the lapse $N = \frac{\ell}{2\rho}$, which correspond to an asymptotic boundary located at $\rho = 0$, the expansion of the boundary metric has the form

$$h_{ij}(\rho,x) = \frac{1}{\rho}\tilde{g}_{ij}(\rho,x) \ , \tag{5.35}$$

$$\begin{aligned}\tilde{g}_{ij}(\rho,x) = g_{(0)ij} + b_{(0)ij}\log\rho + \rho\left(g_{(2)ij} + b_{(2)ij}\log\rho\right) + \\ +\rho^{3/2}\left(g_{(3)ij} + b_{(3)ij}\log\rho\right) + \dots \, .\end{aligned} \tag{5.36}$$

The leading order terms of the series, $g_{(0)ij}$ and $b_{(0)ij}$, respectively, are sources of the dual theory and each one is defining a stress-energy tensor. The coefficient of $b_{(0)ij}$ has to be treated perturbatively, in order for the holographic description to be valid.

5.6.1 Generic boundary geometry

For simplicity, in what follows we choose the unit AdS radius ($\ell = 1$). The inverse boundary metric can be expanded as

$$\tilde{g}^{ij}(\rho,x) = g^{ij}_{(0)} - b^{ij}_{(0)}\log\rho + \rho\tilde{g}^{ij}_{(2)} + \rho^{3/2}\tilde{g}^{ij}_{(3)} + \dots \, , \tag{5.37}$$

where

$$\begin{aligned}\tilde{g}^{ij}_{(2)} &= -g^{ij}_{(2)} - b^{ij}_{(2)}\log\rho + 2\left(b_{(0)}g_{(2)}\right)^{ij}\log\rho + 2\left(b_{(0)}b_{(2)}\right)^{ij}\log^2\rho \, , \\ \tilde{g}^{ij}_{(3)} &= -g^{ij}_{(3)} - b^{ij}_{(3)}\log\rho + 2\left(b_{(0)}g_{(3)}\right)^{ij}\log\rho + 2\left(b_{(0)}b_{(3)}\right)^{ij}\log^2\rho \, .\end{aligned}$$

Here we have to mention, that on the boundary manifold, $g_{(0)}^{ij}$ and $g_{(0)ij}$, are raising and lowering the boundary indices, indicated by Latin letters. In the radial foliation, the extrinsic curvature is defined by the formula $K_{ij} = -\frac{1}{2N}\partial_\rho h_{ij}$, so the asymptotic expansion of $K^i_j = h^{ik}K_{kj}$ in the chosen frame has the form

$$K^i_j = \delta^i_j - b^i_{(0)j} + \rho K^i_{(2)j} + \rho^{3/2}K^i_{(3)j} + \dots , \tag{5.38}$$

where the coefficients of the expansion are given by

$$\begin{aligned} K^i_{(2)j} = -b^i_{(2)j} - g^i_{(2)j} - b^i_{(2)j}\log\rho + \left(b_{(0)}g_{(2)}\right)^i_j + 2\left(b_{(0)}b_{(2)}\right)^i_j\log\rho + \\ + \left(b_{(0)}g_{(2)}\right)^i_j\log\rho + \left(b_{(0)}b_{(2)}\right) , \\ K^i_{(3)j} = -b^i_{(3)j} - \frac{3}{2}g^i_{(3)j} - \frac{3}{2}b^i_{(3)j}\log\rho + \left(b_{(0)}g_{(3)}\right)^i_j + 2\left(b_{(0)}b_{(3)}\right)^i_j\log\rho + \\ + \frac{3}{2}\left(b_{(0)}g_{(3)}\right)^i_j\log\rho + \frac{3}{2}\left(b_{(0)}b_{(3)}\right)^i_j\log^2\rho . \end{aligned}$$

We note that, in contrast to the standard FG expansion (2.13), the relaxed fall-off prevents δK^i_j from vanishing at the leading order, due to the presence of the logarithmic terms. The finiteness of δK at the boundary indicates the presence of a new independent source, $b_{(0)ij}$, apart from the conformal boundary metric, $g_{(0)ij}$. Moreover, the asymptotic structure of the manifold is modified, as it is shown from the asymptotic expansion of the curvature

$$\begin{aligned} R^{i\rho}_{j\rho} &= -\delta^i_j + 2b^i_{(0)j} + \mathcal{O}(\rho) , \\ R^{i\rho}_{jk} &= 2\rho\left(D_k b^i_{(0)j} - D_j b^i_{(0)k}\right) + \mathcal{O}\left(\rho^2\right) , \\ R^{ij}_{kl} &= -\delta^{[ij]}_{[kl]} + b^i_{(0)k}\delta^j_l - b^i_{(0)l}\delta^j_k - b^j_{(0)k}\delta^i_l + b^j_{(0)l}\delta^i_k + \mathcal{O}(\rho) . \end{aligned}$$

The modification of the conformal structure at asymptotic infinity leads the holographic description to break down and the usual AdS/CFT correspondence is no more valid but should be modified, too. Nevertheless, working perturbatively in $b_{(0)ij}$, in the limit that is non-vanishing and sufficiently small, one recovers the asymptotic behavior of AAdS spacetimes. We are interested in the perturbative limit. The dual operator that couples to the new source is the logarithmic stress energy tensor t_{ij}, which is an irrelevant operator that belongs to the LCFT living on the boundary. Namely, the magnitude of t_{ij} decreases when undergoes an RG transformation and as a result is not an observable in the macroscopic scale.

In what follows, we proceed with the explicit calculation of the holographic correlation functions, identified as the tensors coupled to two sources, $\delta g_{(0)ij}$ and

$\delta b_{(0)ij}$, by evaluating Eq.(5.34) in the relaxed FG frame (5.36).

5.6.2 Vanishing log source ($b_{(0)ij} = 0$)

First we derive the energy-momentum tensor of the boundary CFT assuming that the leading order logarithmic term vanishes, for simplicity. As a first step, we are solving the EOM (2.37) in the asymptotic limit using the method of successive approximations and determine the FG coefficients as functions of the intrinsic curvature of the boundary.

Taking the trace of the equation of motion, whose expansion gives ($R = -12$), we get

$$Trg_{(3)} = Trb_{(3)} = 0 = Trb_{(2)}\,, \tag{5.39}$$

$$4Trg_{(2)} + R\left(g_{(0)}\right) = 0\,. \tag{5.40}$$

The (ρi) component of the EOM leads to

$$\nabla_j b^j_{(3)i} = \nabla_j g^j_{(3)i} = \nabla_j b^j_{(2)i} = 0\,, \tag{5.41}$$

$$\nabla_i Trg_{(2)} - \nabla_j g^j_{(2)i} = 0\,. \tag{5.42}$$

For the (ij) part of the EOM, we obtain

$$g^i_{(2)j} - Trg_{(2)}\delta^i_j + R^i_j\left(g_{(0)}\right) - \frac{1}{2}R\left(g_{(0)}\right)\delta^i_j = 0\,, \tag{5.43}$$

$$b^i_{(2)j} = 0\,. \tag{5.44}$$

We conclude that the vanishing of $b_{(0)ij}$ turns $b_{(2)ij}$ to zero in (5.36) but leaves $b_{(3)ij}$ unaffected.

Actually, the absence of the source $b_{(0)ij}$ corresponds to a vanishing logarithmic energy-momentum tensor t_{ij}. Taking into account the above relations, the extrinsic curvature obtains the form

$$K^i_j = \delta^i_j - \rho g^i_{(2)j} + \rho^{3/2}\left(b^i_{(3)j} - \frac{3}{2}g^i_{(3)j} - \frac{3}{2}b^i_{(3)j}\log\rho\right) + \ldots\,. \tag{5.45}$$

Calculating different contributions of the variation of the action (5.34), we find

$$\delta^{[ij]}_{[k\ell]} K^m_i G^\ell_j = 3\rho^{3/2} b^m_{(3)k} , \tag{5.46}$$

$$\frac{1}{N}\delta^{[ij]}_{[k\ell]} \left(\nabla_\rho G^k_i - \nabla_i G^k_\rho\right) = 6\rho^{3/2} b^j_{(3)\ell} , \tag{5.47}$$

$$N\delta^{[ij]}_{[k\ell]} D^k G^\rho_i = 0 . \tag{5.48}$$

Additionally, due to the absence of the logarithmic source $b_{(0)}$, the leading contribution in the variation of the extrinsic curvature is of order $\mathcal{O}(\rho)$. Using the asymptotic expansion of the boundary metric h_{ij}, we determine the fall-off of the following quantities,

$$\left(h^{-1}\delta h\right)^i_j = \left(\tilde{g}^{-1}\delta\tilde{g}\right)^i_j = \left(g^{-1}_{(0)}\delta g_{(0)}\right)^i_j + \mathcal{O}(\rho) \tag{5.49}$$

$$\sqrt{-h} = \frac{\sqrt{-\tilde{g}}}{\rho^{3/2}} = \frac{\sqrt{-g_{(0)}}}{\rho^{3/2}} + \mathcal{O}\left(\rho^{-1/2}\right) . \tag{5.50}$$

Note that the variation of K^i_j is subdominant with respect to the variation of the metric, as it should be in standard AAdS spacetimes. Therefore, the contribution of δK^i_j vanishes at the conformal boundary. Summing all the contributions, the variation of the CrG action (5.34) acquires the form

$$\delta I_{critical} = \frac{9}{32\pi G}\int\limits_{\partial M} d^3x \sqrt{-g_{(0)}} b^i_{(3)j} \left(g^{-1}_{(0)}\delta g_{(0)}\right)^j_i . \tag{5.51}$$

We note that the result is finite and the variational principle is well defined for the Dirichlet boundary condition, $\delta g_{(0)ij} = 0$. Hence, the CrG action (5.9) is finite and no additional counterterms are needed, in the absence of the logarithmic source.

According to the Ref.[2], the holographic stress tensor is defined as the functional variation of the regular part of the surface term with respect to the source,

$$\langle T_{ij} \rangle = -\frac{2}{\sqrt{-g_{(0)}}} \frac{\delta I}{\delta g^{ij}_{(0)}} , \tag{5.52}$$

which applied in (5.51), becomes

$$\langle T_{ij} \rangle = \frac{9}{16\pi G} b_{(3)ij} . \tag{5.53}$$

This formula is in agreement with the result in Ref.[53], generalized for an arbitrary form of the boundary geometry. When Einstein spaces are taken into account, the logarithmic contribution $b_{(3)}$ vanishes and $\left\langle T_{(ij)} \right\rangle$ turns to zero, recovering the results of Ref.[24].

5.7 Linearized analysis ($b_{(0)ij} \neq 0$)

Switching on the logarithmic source makes the calculation of the holographic correlation functions cumbersome. In order to simplify the derivation, we are working perturbatively around AdS_4 [53]–[55].

In the linearization limit, the on-shell action contains terms up to quadratic order in the perturbation, what is sufficient for the derivation of the two-point correlation functions at the boundary. In this section, starting from the new formulation of the CrG action (5.18), we evaluate the linearized AdS_4 metric and regularize the action introducing proper counterterms.

In order to technically simplify the derivation of the correlation functions, we consider a flat boundary metric. Thus, we are evaluating the linearized Lagrangian (5.34) in Gauss-normal coordinates (5.28), where the regular boundary metric $g_{ij}(\rho, x)$ is written as a deviation $c_{ij}(\rho, x)$ of the Minkowski background,

$$h_{ij}(\rho, x) = \frac{1}{\rho}\tilde{g}_{ij} = \frac{1}{\rho}\left(\eta_{ij} + c_{ij}(\rho, x)\right) . \tag{5.54}$$

The perturbation $c_{ij}(\rho, x)$ is a regular function on the boundary, so it admits a series expansion around the flat background of the form

$$\begin{aligned} c_{ij} = h_{(0)ij} + b_{(0)ij}\log\rho + \rho\left(g_{(2)ij} + b_{(2)ij}\log\rho\right) + \\ +\rho^{3/2}\left(g_{(3)ij} + b_{(3)ij}\log\rho\right) + \dots , \end{aligned} \tag{5.55}$$

where $h_{(0)ij}$ corresponds to the source of CFT stress-energy tensor $\langle T_{ij}\rangle$ and $b_{(0)ij}$ is the logarithmic source dual to another stress tensor $\langle t_{ij}\rangle$.

As a first step, we determine the relations between the FG coeeficients by solving the EOM (2.37) in the FG frame.

From the Ricci scalar $R = -12$, we obtain

$$\mathrm{Tr}b_{(0)} = \mathrm{Tr}b_{(3)} = \mathrm{Tr}g_{(3)} = 0 , \tag{5.56}$$

$$4\mathrm{Tr}b_{(2)} + \partial_i\partial_j b^{ij}_{(0)} = 0 , \tag{5.57}$$

$$4\mathrm{Tr}g_{(2)} + \partial_i\partial_j h^{ij}_{(0)} - \partial^m\partial_m \mathrm{Tr}h_{(0)} = 0 . \tag{5.58}$$

For the $(\rho\rho)$ component of the EOM (2.37), one gets

$$4\mathrm{Tr}b_{(2)} - \partial_i\partial_j b^{ij}_{(0)} = 0\,, \tag{5.59}$$

while the tracelessness of the Bach tensor gives

$$2\mathrm{Tr}b_{(2)} - \frac{7}{2}\partial_i\partial_j b^{ij}_{(0)} = 0\,. \tag{5.60}$$

Combining these relations with Eq. (5.57), leads to

$$\mathrm{Tr}b_{(2)} = \partial_i\partial_j b^{ij}_{(0)} = 0\,. \tag{5.61}$$

For the (ρi) components of the EOM terms we get

$$\partial_j b^j_{(0)i} = \partial_j b^j_{(2)i} = \partial_j b^j_{(3)i} = 0\,, \tag{5.62}$$

$$4\partial_j g^j_{(2)i} + \partial_i\partial_m\partial_k h^{mk}_{(0)} - \partial_i\partial^m\partial_m \mathrm{Tr}h_{(0)} = 0\,. \tag{5.63}$$

Finally, the (ij) part of the EOM reads

$$\partial^m\partial_m b^i_{(0)j} = 2b^i_{(2)j}\,, \tag{5.64}$$

$$\left(D^2 h_{(0)}\right)^i_j = 2g^i_{(2)j} + 2\delta^i_j \mathrm{Tr}g_{(2)} - 8b^i_{(2)j}\,, \tag{5.65}$$

where

$$\left(D^2 g_{(n)}\right)_{ij} = \partial_i\partial_j\left(\mathrm{Tr}g_{(n)}\right) + \partial^m\partial_m g_{(n)ij} - \left(\partial_i\partial_k g^k_{(n)j} + \partial_j\partial_k g^k_{(n)i}\right)\,, \tag{5.66}$$

corresponds to the D^2 operator, introduced in Ref.[53]. Here, the covariant derivatives have been exchanged to partial ones as they are defined with respect to the background Minkowski metric η_{ij}.

In what follows, we determine different terms in the variation of the action (5.34) by evaluating the AdS$_4$ metric through its asymptotic expansion given in (5.54) and (5.55), respectively.

First, the determinant and the variation of the metric are expanded as

$$\sqrt{-h} = \frac{\sqrt{-\tilde{g}}}{\rho^{3/2}} = \frac{1}{\rho^{3/2}}\left(1 + \frac{1}{2}\mathrm{Tr}c\right)\,, \tag{5.67}$$

$$\left(h^{-1}\delta h\right)^i_j = \left(\tilde{g}^{-1}\delta\tilde{g}\right)^i_j = \left(\eta^{im} - c^{im}\right)\delta c_{mj}\,. \tag{5.68}$$

Furthermore, the coefficient of δK in (5.34) is expanded as

$$\delta^{[ij]}_{[k\ell]} G^{\ell}_{j} = -3b^{i}_{(0)k} - 3\rho b^{i}_{(2)k} + 3\rho^{3/2} b^{i}_{(3)k} \,. \tag{5.69}$$

The second part refers to the coefficients of the variation of the metric, which contains contributions from the following quantities,

$$\begin{aligned}\delta^{[ij]}_{[k\ell]} K^{m}_{i} G^{\ell}_{j} &= -3b^{m}_{(0)k} + 3\rho\left[2\left(b_{(0)}b_{(2)}\right)^{m}_{k} - b^{m}_{(2)k} + \left(b_{(0)}g_{(2)}\right)^{m}_{k} + \left(b_{(0)}b_{(2)}\right)^{m}_{k}\log\rho\right] \\ &+ 3\rho^{3/2}\left[b^{m}_{(3)k} + \frac{3}{2}\left(b_{(0)}g_{(3)}\right)^{m}_{k} + \frac{3}{2}\left(b_{(0)}b_{(3)}\right)^{m}_{k}\log\rho\right] ,\end{aligned}$$

and

$$\delta^{[ij]}_{[k\ell]} \frac{1}{N}\left(\nabla_{\rho} G^{k}_{i} - \nabla_{i} G^{k}_{\rho}\right) = 3b^{j}_{(0)\ell} + 3\rho\,\delta I^{(2)}_{critical} + \frac{9}{2}\rho^{3/2}\delta I^{(3)}_{critical} \,, \tag{5.70}$$

where

$$\begin{aligned}\delta I^{(2)}_{critical} &= -2\left(b_{(0)}b_{(2)}\right)^{j}_{\ell} - b^{j}_{(2)\ell} - \left(b_{(0)}g_{(2)}\right)^{j}_{\ell} + \\ &+\delta^{j}_{\ell}\left(2\mathrm{Tr} b_{(0)}b_{(2)} + \mathrm{Tr} b_{(0)}g_{(2)} + \mathrm{Tr} b_{(0)}b_{(2)}\log\rho\right) - \left(b_{(0)}b_{(2)}\right)^{j}_{\ell}\log\rho \,,\end{aligned}$$

and

$$\delta I^{(3)}_{critical} = \frac{4}{3} b^{j}_{(3)\ell} + \delta^{j}_{\ell}\mathrm{Tr} b_{(0)}g_{(3)} - \left(b_{(0)}g_{(3)}\right)^{j}_{\ell} - \left(b_{(0)}b_{(3)}\right)^{j}_{\ell}\log\rho + \mathrm{Tr} b_{(0)}b_{(3)}\delta^{j}_{\ell}\log\rho \,.$$

The term $D^{k}G^{\rho}_{i}$ of (5.34) vanishes in the linearization limit. Here, the terms $b_{(0)}b_{(2)}, b_{(0)}g_{(2)}, b_{(0)}b_{(3)}$ and $b_{(0)}g_{(3)}$ are of order $\mathcal{O}\left(c^{2}\right)$ and they are dropped from the derivation, as they contribute to three-point correlation functions that are not included in our study.

The terms that contribute up to quadratic order in c_{ij} in Eq.(5.34) adopt the form

$$\begin{aligned}\delta I_{critical} = \frac{1}{32\pi G}\int_{\partial M} d^{3}x \Big(&6\rho^{-3/2} b_{(0)ij}\delta b^{ij}_{(0)} - 6b_{(3)ij}\delta b^{ij}_{(0)} + \\ &+ 9b_{(3)ij}\log\rho\,\delta b^{ij}_{(0)} + 9b_{(3)ij}\delta h^{ij}_{(0)}\Big) \,,\end{aligned} \tag{5.71}$$

where the order $\mathcal{O}\left(\rho^{-1/2}\right)$ divergences have been dropped as well, as they are linear in $b_{(2)ij}$, which are total derivatives, as shown in the field equations (5.64).

Note that the variation of the action contains divergent terms, which correspond to the logarithmic stress energy tensor. On the other hand, the holographic conjugate of the Einstein source $h_{(0)ij}$ is finite. As a consequence, no divergences should be considered when the Einstein modes are taken into account. Rendering the action finite is equivalent to regularizing the non-Einstein branch of the theory.

Towards this objective, we track down the divergences associated to the logarithmic part at the level of the action, using HR. The first step towards the regularization of the CrG action (5.18), is to evaluate it on-shell, acquiring the form

$$I_{critical} = -\frac{1}{32\pi G}\int\limits_{M} d^4x\sqrt{-g}G^{\mu}_{\nu}G^{\nu}_{\mu}\,. \tag{5.72}$$

Considering the linearized EOM (5.56-5.65) and after some algebraic manipulation, the square of the Einstein tensor is written as

$$G^{\mu}_{\nu}G^{\nu}_{\mu} = 9\mathrm{Tr}b^2_{(0)} + 18\rho\mathrm{Tr}b_{(0)}b_{(2)} - 18\rho^{3/2}\mathrm{Tr}b_{(0)}b_{(3)}\,. \tag{5.73}$$

We are introducing a cutoff scale at finite radius $\rho = \varepsilon$, in order to track down the divergences. In this case, the action (5.72) can be cast in the form,

$$\begin{aligned} I_{critical} &= -\frac{1}{64\pi G}\int d^3x\int_{\rho=\varepsilon} d\rho\frac{\sqrt{-\tilde{g}}}{\rho^{3/2+1}}G^{\mu}_{\nu}G^{\nu}_{\mu} \\ &= -\frac{9}{64\pi G}\int d^3x\int_{\rho=\varepsilon} d\rho\frac{\sqrt{-\tilde{g}}}{\rho^{3/2+1}}\left(\mathrm{Tr}b^2_{(0)} + 2\rho\mathrm{Tr}b_{(0)}b_{(2)} - 2\rho^{3/2}\mathrm{Tr}b_{(0)}b_{(3)}\right) \\ &= \frac{9}{32\pi G}\int\limits_{\partial M} d^3x\left(\mathrm{Tr}b_{(0)}b_{(3)}\log\varepsilon + \frac{1}{3}\varepsilon^{-3/2}\mathrm{Tr}b^2_{(0)} + 2\varepsilon^{-1/2}\mathrm{Tr}b_{(0)}b_{(2)}\right)\,. \end{aligned} \tag{5.74}$$

It is evident that all the terms tend to infinity when evaluated at the conformal boundary ($\varepsilon = 0$), giving rise to the infinities encountered before at the level of the variation of the action.

5.8 Counterterms

Rendering the action finite requires the addition of proper surface terms, called counterterms, on top of the CrG action. The terms (5.74) must be added to the action with the opposite sign to cancel divergences and constitute the counterterms when expressed in terms of boundary quantities such that the variational problem is well-defined. There are two criteria that have to be satisfied by the varied action: i) finiteness, and ii) Dirichlet boundary condition of the sources.

As a first step towards the derivation of the counterterms, we invert the series given by Eq.(5.55) as follows

$$b_{(0)ij} = \rho\partial_\rho c_{ij} - \rho\left(b_{(2)ij} + g_{(2)ij} + b_{(2)ij}\log\rho\right) - \rho^{3/2}\left(g_{(3)ij} + b_{(3)ij}\log\rho\right) .$$

The following combination

$$\frac{1}{3}\rho^{1/2}\partial_\rho c_{ij}\partial_\rho c^{ij} = \frac{2}{3}\mathrm{Tr}b_{(0)}b_{(3)} + \mathrm{Tr}b_{(0)}g_{(3)} + \mathrm{Tr}b_{(0)}b_{(3)}\log\rho + \frac{1}{3}\rho^{-3/2}\mathrm{Tr}b_{(0)}^2 + \frac{2}{3}\rho^{-1/2}\left(\mathrm{Tr}b_{(0)}b_{(2)} + \mathrm{Tr}b_{(0)}g_{(2)} + \mathrm{Tr}b_{(0)}b_{(2)}\log\rho\right) , \quad (5.75)$$

recovers the terms in (5.74) but introduces new infinities of order $\mathcal{O}\left(\rho^{-1/2}\right)$ up to a finite contribution. The trace of the subdominant term $b_{(0)}g_{(2)}$ can be rewritten as

$$\begin{aligned} \mathrm{Tr}b_{(0)}g_{(2)} = b_{(0)}^{ij}g_{(2)ij} &= \frac{1}{2}b_{(0)}^{ij}\left(D^2h_{(0)}\right)_{ij} - \mathrm{Tr}b_{(0)}\mathrm{Tr}g_{(2)} + 4b_{(0)}^{ij}b_{(2)ij} \\ &= \frac{1}{2}h_{(0)}^{ij}\partial^m\partial_m b_{(0)ij} + 4\mathrm{Tr}b_{(0)}b_{(2)} \\ &= \mathrm{Tr}h_{(0)}b_{(2)} + 4\mathrm{Tr}b_{(0)}b_{(2)} , \end{aligned} \quad (5.76)$$

where the Eqs.(5.64) and (5.65) were taken into account. For the transition from the second to the third line, integration by parts was considered. Moreover, the following expressions are valid

$$c^{ij}\partial^m\partial_m\partial_\rho c_{ij} = 2\rho^{-1}\left(\mathrm{Tr}h_{(0)}b_{(2)} + \mathrm{Tr}b_{(0)}b_{(2)}\log\rho\right) + \mathcal{O}\left(\rho^0\right) , \quad (5.77)$$

$$\partial_\rho c^{ij}\partial^m\partial_m\partial_\rho c_{ij} = 2\rho^{-2}\mathrm{Tr}b_{(0)}b_{(2)} + \mathcal{O}\left(\rho^{-1}\right) , \quad (5.78)$$

after inverting the series. Here, we note that there is a specific linear combination of the Eqs.(5.75)- (5.78) of the form

$$\frac{1}{3}\rho^{1/2}\left(\partial_\rho c_{ij}\partial_\rho c^{ij} - c^{ij}\partial^m\partial_m\partial_\rho c_{ij} - 2\rho\partial_\rho c^{ij}\partial^m\partial_m\partial_\rho c_{ij}\right) =$$
$$= \mathrm{Tr}b_{(0)}b_{(3)}\log\rho + 2\rho^{-1/2}\mathrm{Tr}b_{(0)}b_{(2)} + \frac{1}{3}\rho^{-3/2}\mathrm{Tr}b_{(0)}^2 + \frac{2}{3}\mathrm{Tr}b_{(0)}b_{(3)} + \mathrm{Tr}b_{(0)}g_{(3)} ,$$

that recovers all the terms of Eq.(5.74) up to finite terms. Hence, after fixing the coupling constant, the counterterms acquire the form

$$I_{ct} = -\frac{3}{32\pi G}\int_{\partial M} d^3x\rho^{1/2}\left(\partial_\rho c_{ij}\partial_\rho c^{ij} - c^{ij}\partial^m\partial_m\partial_\rho c_{ij} - 2\rho\partial_\rho c^{ij}\partial^m\partial_m\partial_\rho c_{ij}\right) . \quad (5.79)$$

Finally, in order to have a covariant form of counterterms the proper rescaling of the boundary metric have to be considered, namely $h_{ij} = \left(\eta_{ij} + c_{ij}\right) / \rho$. Additionally, the extrinsic curvature is given by $K_{ij} = \frac{1}{\rho}\eta^i_j - \kappa_{ij}$, where $\kappa_{ij} = \partial_\rho c_{ij}$. Applying the respective rescalings, the covariant form of the counterterms action read

$$I_{ct} = \frac{3}{32\pi G} \int_{\partial M} d^3x \sqrt{-h} \left(2K - K_{ij}K^{ij} + \frac{1}{N} K^{ij} \Box K_{ij} - \frac{1}{2N} \Box K - 3 \right) . \qquad (5.80)$$

Unlike the counterterms in Ref.[53], in our formulation there is an explicit dependence on extrinsic counterterms. The difference stems from the different boundary conditions that are imposed in the problem. In the latter reference, Dirichlet boundary conditions for the induced boundary metric h_{ij} was used, instead of the conformal boundary metric $g_{(0)ij}$. Another interesting property is that I_{ct} cancels only the divergences coming from non-Einstein part of the action, as the Einstein modes are renormalized thanks to the presence of the topological invariant (GB) in the CrG action (5.10).

5.9 Properties of Logarithmic CFTs

In this section we review some of the main properties of the LCFTs, which will play a significant role in the analysis of the next section.

Even though the presence of logarithms in CFTs was not considered in the seminal papers of CFT during the 80's, it was shown in [56] that the presence of logarithms does not violate neither the scale nor the conformal invariance. Later [43], it was introduced the concept of logarithmic operators that generalizes primary operators in the case of Jordan block matrices. This new class of operators reproduces the logarithmic structure of the theory that can be recognized even at the level of the Virasoro algebra.

In general there are a lot of condensed matter systems that is believed to be described by LCFTs. They usually to CFTs with a central charge equal to zero. They are encountered in disordered systems [44], percolation, polymers [57], [58], turbulence [59], the Quantum Hall effect [60], string theory [61] etc.

Using the state/operator map, it is convenient to discuss initially for the behavior of the logarithmic states under the action of the Virasoro operators. Logarithmic states are annihilated by all L_n with $n > 0$, but contrary to the common primary states, L_0 is not a diagonalizable matrix. More explicitly,

$$L_0|C\rangle = h|C\rangle, L_0|D\rangle = h|D\rangle + |C\rangle . \qquad (5.81)$$

Notice the Jordan block structure of L_0. The state $|C\rangle$ looks like a primary state but this is no longer true for $|D\rangle$. In this configuration $|D\rangle$ is considered as the logarithmic partner of $|C\rangle$. Considering that L_0 is associated to the Hamiltonian, as it corresponds to the time translation generator, the non-diagonalizability means that it is no longer an Hermitian matrix, leading to non-unitary theories.

Using the state/operator map, we can define the operators $T(z)$ and $t(z)$ for the states $|C\rangle$ and $|D\rangle$, respectively. Demanding global conformal invariance, we define uniquely the form of the two-point correlation functions as

$$\langle T(z_1)T(z_2)\rangle = 0\,, \tag{5.82}$$

$$\langle T(z_1)t(z_2)\rangle = \langle t(z_1)T(z_2)\rangle = \frac{B}{(z_1-z_2)^{2h}}\,, \tag{5.83}$$

$$\langle t(z_1)t(z_2)\rangle = \frac{-2B\ln(z_1-z_2)+a}{(z_1-z_2)^{2h}}\,. \tag{5.84}$$

Notice the presence of the logarithm in the two-point function of the operator $D(z)$, justifying its name as a logarithmic operator. Moreover, the values of the constants a and b cannot be fixed by the conformal invariance, leaving a degree of arbitrariness in the determination of $D(z)$ in Eq.(5.81). Actually, the constant a can be removed by imposing a field redefinition of $D \to D + \gamma C$, where γ is an arbitrary constant. Finally, it is shown that in LCFTs the primary countertpart of the logarithmic operator is has zero norm, as it can be seen form the two-point function of $C(z)$.

Logarithmic operators origin can be found in the operator product expansion (OPE) of a certain class of primary operators. Indeed, in case of primary operators that belong to the Kac table emerge logarithmic singularities in the differential equation satisfied by the four point function. As shown in [62], for a primary operator $A(z)$ with a four point function

$$\langle A(z_1)A(z_2)A(z_3)A(z_4)\rangle = \frac{1}{(z_1-z_3)^{2h}(z_2-z_4)^{2h}}F(x)\,, \tag{5.85}$$

where h is the conformal dimension and x reads

$$x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}\,. \tag{5.86}$$

The aforementioned logarithmic singularities that arise in $F(x)$ can be explained when logarithmic terms are implemented in the OPE of $A(z)$ such that

$$A(z)A(0) \sim z^{\alpha}\left(C(0)\ln z + D(0)\right) + \ldots\,, \tag{5.87}$$

where α is the leading order power in the logarithmic singularity of $F(x)$ as shown below

$$F(x) \sim x^{\alpha}\ln x\,. \tag{5.88}$$

The pair of operators C and D that arise in the OPE, obtain the same two-point functions structure as in Eqs.(5.84). As a results they correspond to a logarithmic pair where C is the primary operator and D is the logarithmic counterpart. The reason behind this behavior, is the degeneracy of the conformal dimension as seen in 5.87.

The power α denotes the conformal dimension of the primary operators arising in the RHS of the OPE expansion, i.e., $\delta = \alpha + 2h$. The presence of two operators with coinciding conformal dimension is the reason that gives rise to the logarithms.

In the following section we are seeking for the consequences of the degeneracies arising in the context of the Critical Gravity, as massive modes turn to massless, in the boundary CFT and its possibility to be classified as an LCFT.

5.10 Holographic correlation functions

Having defined a new set of counterterms, we ensured the finiteness of the action and we are now able to compute the correlation functions at the boundary CFTs.

First we have to compute the variation of the total action $I_{tot} = I_{CrG} + I_{ct}$. It consists of two parts. The first part corresponds to the variation of the linearized version of the CrG action δI_{CrG}, given in Eq.(5.71), while the second part refers to the variation of the counterterms δI_{ct}, which have the form

$$
\begin{aligned}
\delta I_{ct} &= -\frac{3}{32\pi G}\int_{\partial M} d^3x\rho^{1/2}\left[-\partial^m\partial_m\partial_\rho c_{ij}\delta c^{ij} + \left(2\partial_\rho c_{ij} - \partial_m\partial^m c_{ij} - 4\rho\partial^m\partial_m\partial_\rho c_{ij}\right)\delta\left(\partial_\rho c^{ij}\right)\right] \\
&= -\frac{3}{16\pi G}\int_{\partial M} d^3x\rho^{1/2}\partial_\rho c_{ij}\delta\left(\partial_\rho c^{ij}\right) . \qquad (5.89)
\end{aligned}
$$

The total derivative terms have been dropped because ∂M is without boundary. Summing up these two contributions, we get

$$
\begin{aligned}
\delta I_{total} &= \frac{1}{32\pi G}\int_{\partial M} d^3x\left(6\rho^{-3/2}b_{(0)ij}\delta b^{ij}_{(0)} - 6b_{(3)ij}\delta b^{ij}_{(0)} + 9b_{(3)ij}\log\rho\delta b^{ij}_{(0)} + 9b_{(3)ij}\delta h^{ij}_{(0)}\right. \\
&\quad \left. -6\rho^{-3/2}b_{(0)ij}\delta b^{ij}_{(0)} - 6b_{(3)ij}\delta b^{ij}_{(0)} - 9b_{(3)ij}\log\rho\delta b^{ij}_{(0)} - 9g_{(3)ij}\delta b^{ij}_{(0)}\right) \\
&= \frac{1}{32\pi G}\int_{\partial M} d^3x\left(-12b_{(3)ij}\delta b^{ij}_{(0)} - 9g_{(3)ij}\delta b^{ij}_{(0)} + 9b_{(3)ij}\delta h^{ij}_{(0)}\right) . \qquad (5.90)
\end{aligned}
$$

Now the derivation of the holographic stress energy tensor, defined as the functional derivative in the sources, is straightforward. The one-point function, dual to the source $h_{(0)ij}$, reads

$$\langle T_{ij} \rangle = 2\frac{\delta I_{total}}{\delta h^{ij}_{(0)}} = \frac{9}{16\pi G} b_{(3)ij} , \tag{5.91}$$

while the dual to the source $b_{(0)ij}$ is given by

$$\langle t_{ij} \rangle = 2\frac{\delta I_{total}}{\delta b^{ij}_{(0)}} = -\frac{3}{16\pi G}\left(4b_{(3)ij} + 3g_{(3)ij}\right) . \tag{5.92}$$

The two-point correlation functions are defined as the variations of the above correlators with respect to the sources. More specifically, taking into account the presence of two sources, the following correlators can be defined

$$\langle T_{ij}(x)\, T_{kl}(x') \rangle = -2i\frac{\delta}{\delta h^{kl}_{(0)}(x')}\langle T_{ij}(x)\rangle = -\frac{9i}{8\pi G}\frac{\delta b_{(3)ij}(x)}{\delta h^{kl}_{(0)}(x')} = 0 , \tag{5.93}$$

$$\begin{aligned}\langle T_{ij}(x)\, t_{kl}(x') \rangle &= -2i\frac{\delta}{\delta b^{kl}_{(0)}(x')}\langle T_{ij}(x)\rangle = -2i\frac{\delta}{\delta h^{kl}_{(0)}(x')}\langle t_{ij}(x)\rangle \\ &= -\frac{9i}{8\pi G}\frac{\delta b_{(3)ij}(x)}{\delta b^{kl}_{(0)}(x')} = \frac{9i}{8\pi G}\frac{\delta g_{(3)ij}(x)}{\delta h^{kl}_{(0)}(x')} ,\end{aligned} \tag{5.94}$$

$$\langle t_{ij}(x)\, t_{kl}(x') \rangle = -2i\frac{\delta}{\delta b^{kl}_{(0)}(x')}\langle t_{ij}(x)\rangle = \frac{3i}{8\pi G}\left(4\frac{\delta b_{(3)ij}(x)}{\delta b^{kl}_{(0)}(x')} + 3\frac{\delta g_{(3)ij}(x)}{\delta b^{kl}_{(0)}(x')}\right) . \tag{5.95}$$

The i factor in the two-point point functions arises due to Lorentzian signature of spacetime. The origin of the factor can be found in the relation between the generating functional of the boundary CFT and the bulk on-shell action, which for the Lorentzian signature can be written as $W_L \sim iI_L$. This choice yields the formulas displayed above [55].

For the explicit derivation of the two-point correlation functions a detailed analysis of the mode structure of the theory is needed. The mode analysis of CrG has been given in Ref.[53], where it is clear that all the logarithmic modes are independent of the source $h_{(0)ij}$. As a consequence the norm of the standard stress-energy tensor T_{ij} is zero, in accordance to the correlators structure in a LCFT.

The inverse is not true, as there are non-logarithmic modes whose source is $b_{(0)ij}$. This is a property that can be deduced directly from the EOM (5.56-5.62). There it is shown that $b_{(3)ij}$ is a transverse ($\partial_j b^j_{(3)i} = 0$) and traceless mode whereas $g_{(3)ij}$ it is just traceless. This feature of $g_{(3)ij}$ is not valid in Einstein gravity, where it is both transverse and traceless and determines the holographic stress-energy tensor. The deviation of $g_{(3)ij}$ from traversality in CrG, indicates the presence of non-Einstein modes and expresses the deviation from the Einstein branch of the theory. Hence, $g_{(3)ij}$ depends explicitly on $b_{(0)ij}$, which is the source of the non-Einstein modes. As a consequence of its richer structure, $g_{(3)}$ is decomposed to irreducible components as

$$g_{(3)ij} = \nabla_i V_j^{(3)} + \nabla_j V_i^{(3)} + g_{(3)ij}^{TT} + \left(\nabla_i \nabla_j - \frac{1}{3}\eta_{ij}\nabla^2\right) S^{(3)} . \tag{5.96}$$

Different parts of this decomposition are decoupled in the EOM and contribute independently in the one-point function t_{ij} which consists of: i) a transverse vector V_i, ii) a transverse traceless part t_{ij}^{TT}, which is the logarithmic conjugate of T_{ij}, iii) and a scalar S. Here, t_{ij}^{TT} is the logarithmic conjugate of T_{ij} and corresponds to the logarithmic stress-energy tensor of the LCFT.

Considering the Eq.(5.92), the three operators can be cast in the form:

$$\langle t_{ij}^{TT} \rangle = -\frac{3}{16\pi G}\left(4b_{(3)ij} + 3g_{(3)ij}^{TT}\right) , \tag{5.97}$$

$$\langle V_i \rangle = -\frac{9}{16\pi G} V_i^{(3)} , \tag{5.98}$$

$$\langle S \rangle = -\frac{9}{16\pi G} S^{(3)} . \tag{5.99}$$

Due to its transverse traceless (TT) property, the $b_{(3)ij}$ mode contributes only to the t_{ij}^{TT} operator, leading to an explicit dependence of both the vector and the scalar operator from $g_{(3)ij}$.

Having computed all the operators defining the one-point functions of the theory, we are able to continue with the computation of the two-point correlators. It is useful to mention that the EOM indicate that not only $b_{(3)ij}$ but also the logarithmic source $b_{(0)ij}$ is transverse and traceless. Thus, the following decomposition is valid

$$\frac{\delta g_{(3)ij}}{\delta b_{(0)}^{kl}} = \left(\frac{\delta g_{(3)ij}}{\delta b_{(0)}^{kl}}\right)_V + \left(\frac{\delta g_{(3)ij}}{\delta b_{(0)}^{kl}}\right)_S + \left(\frac{\delta g_{(3)ij}}{\delta b_{(0)}^{kl}}\right)_{TT} . \tag{5.100}$$

where

$$
\begin{aligned}
\left(\frac{\delta g_{(3)ij}}{\delta b^{kl}_{(0)}}\right)_V &= \frac{2i}{9|p|} p_{(i}\Theta_{j)(l}p_{k)}\,, \\
\left(\frac{\delta g_{(3)ij}}{\delta b^{kl}_{(0)}}\right)_{TT} &= (C - 2\log|p|)\,\frac{\delta g_{(3)ij}}{\delta h^{kl}_{(0)}}\,, \\
\left(\frac{\delta g_{(3)ij}}{\delta b^{kl}_{(0)}}\right)_S &= -\frac{i}{12|p|}\left(p_i p_j - \frac{p^2}{3}\eta_{ij}\right)\left(p_k p_l - \frac{p^2}{3}\eta_{kl}\right).
\end{aligned}
$$

Here, p_i is the momentum of the propagating mode, C is a numerical constant and $\Theta = \eta_{ij}p^2 - p_i p_j$.

Indeed, the only non-vanishing mixed correlator in Eq.(5.94) is the one between the two stress-energy tensors. Deriving with respect to $b_{(0)ij}$ means that only operators with a TT component can survive. Given the mode dependence on the sources in [53], we obtain that

$$
\langle T_{ij}(x)\, t^{TT}_{kl}(0)\rangle = -\frac{1}{2\pi^3}\frac{3}{2G}\hat{\Delta}_{ij,kl}\frac{1}{|x^2|}\,, \tag{5.101}
$$

where we defined the differential operator

$$
\hat{\Delta}_{ij,kl} = \frac{1}{2}\left(\hat{\Theta}_{ik}\hat{\Theta}_{jl} + \hat{\Theta}_{il}\hat{\Theta}_{jk} - \hat{\Theta}_{ij}\hat{\Theta}_{kl}\right), \tag{5.102}
$$

$$
\hat{\Theta}_{ij} = \partial_i\partial_j - \eta_{ij}\Box\,, \tag{5.103}
$$

with $\hat{\Theta}_{ij}$ being the Fourier transform of Θ_{ij}. Finally, for the tt correlators (5.95), using the non-vanishing functional derivatives (5.100) and the mode analysis in Ref.[53], we get three independent correlators, corresponding to the vector, scalar and TT operators.

The former ones, when computed, give

$$
\begin{aligned}
\langle V_i(x)\, V_j(0)\rangle &= \frac{9i}{8\pi G}\left(\frac{\delta g_{(3)ij}(x)}{\delta b^{kl}_{(0)}(0)}\right)_V = \frac{1}{2\pi^3}\frac{9i}{8\pi G}\int d^3p e^{ipx}\left(\frac{\delta g_{(3)ij}}{\delta b^{kl}_{(0)}}(p)\right)_V \\
&= -\frac{1}{2\pi^3}\frac{1}{4G}\hat{\Theta}_{ij}\frac{1}{|x^2|}\,,
\end{aligned} \tag{5.104}
$$

and

$$\begin{aligned}
\langle S(x)\,S(0)\rangle &= \frac{9i}{8\pi G}\left(\frac{\delta g_{(3)ij}(x)}{\delta b^{kl}_{(0)}(0)}\right)_S = \frac{1}{2\pi^3}\frac{9i}{8\pi G}\int d^3p e^{ipx}\left(\frac{\delta g_{(3)ij}}{\delta b^{kl}_{(0)}}(p)\right)_S \\
&= \frac{1}{2\pi^3}\frac{3}{8G}\frac{1}{|x^2|}\,.
\end{aligned} \tag{5.105}$$

The latter two-point function is the one corresponding to the logarithmic stress-energy tensors. It receives contributions from the TT part of both $b_{(3)}$ and $g_{(3)}$. An equivalent form of the Eq.(5.95) is

$$\langle t^{TT}_{ij}(x)\,t^{TT}_{kl}(x')\rangle = -\frac{4}{3}\langle T_{ij}(x)\,t^{TT}_{kl}(x')\rangle + \frac{9i}{8\pi G}\left(\frac{\delta g_{(3)ij}(x)}{\delta b^{kl}_{(0)}(x')}\right)_{TT}. \tag{5.106}$$

In this case the correlator acquires the form

$$\begin{aligned}
\langle t^{TT}_{ij}(x)\,t^{TT}_{kl}(0)\rangle &= -\frac{4}{3}\langle T_{ij}(x)\,t^{TT}_{kl}(0)\rangle + \frac{1}{2\pi^3}\frac{9i}{8\pi G}\int d^3p e^{ipx}\left(\frac{\delta g_{(3)ij}}{\delta b^{kl}_{(0)}}(p)\right)_{TT} \\
&= -\frac{1}{2\pi^3}\frac{3}{2G}\hat{\Delta}_{ij,kl}\frac{\log|x^2| + C + 4\gamma - 4/3}{|x^2|}\,,
\end{aligned} \tag{5.107}$$

where C is a real constant. Generally speaking, the logarithmic stress tensor is defined up to addition of a multiple of $\langle T_{ij}\rangle$. Therefore, taking advantage of this freedom, we redefine t^{TT}_{ij} as $t^{TT}_{ij} \to -(C/4+\gamma-1/3)\,T_{ij}$, canceling all the numerical constants appearing in the numerator.

The final result for the two point correlation function of the logarithmic stress energy tensor is

$$\langle t^{TT}_{ij}(x)\,t^{TT}_{kl}(0)\rangle = -\frac{1}{2\pi^3}\frac{3}{2G}\hat{\Delta}_{ij,kl}\frac{\log|x^2|}{|x^2|}\,. \tag{5.108}$$

The vanishing norm of the stress-energy tensor T_{ij} and the logarithmic dependence arising in the norm of its logarithmic conjugate t_{ij}, is a characteristic property of the LCFTs. In general, the correlation functions derived recover the structure of the norm states in LCFT, demonstrating the validity of the new formulation of CrG.

Chapter 6

Conclusions

In this thesis, we explored consequences of a new regularization scheme, called the Kounterterms, gaining an important insight about the properties and holographic applications of higher-derivative theories of gravity in 4D. Starting in Chapter 2, we compare the Kounterterms with the standard counterterms series, introduced in Holographic Renormalization, showing that there is full matching between the two schemes up to 8th-derivative terms in the boundary, for ACF spaces. The equivalence is highly non-trivial, considering the fact that the Kounterterms depend on the extrinsic curvature while the counterterms are functions of intrinsic quantities of the boundary. We derived a closed and recursive formula for the counterterms which turned out to be common for both even- and odd-dimensional manifolds. For even-dimensional manifolds, the Kounterterms can be obtained from a topological invariant of the Euler class of the corresponding dimension, demonstrating that the addition of a topological invariant with a fixed coupling constant is equivalent to the Holographic Renormalization.

The corresponding topological invariant in 4D is the Gauss-Bonnet term and when added to the Einstein-Hilbert action, the total action acquires the form of the MacDowell-Mansouri action for the AdS_4 group. In Chapter 3, we showed its equivalence with CG for Bach-flat spacetimes, when the CG coupling constant is fixed. The argument consists in switching on the trace-free part of the Ricci tensor, corresponding to a more generic class of spacetimes than the Einstein ones. The decomposition splits the Weyl tensor in two parts, whose non-Einstein part is identified by terms linear in the Bach tensor. Switching off the non-Einstein modes requires the vanishing of the Bach tensor and imposing adequate Neumann boundary conditions. This argumentation provides an explicit proof on the equivalence between CG and Einstein gravity with a negative cosmological constant.

The curvature decomposition discussed above allows us to identify the non-Einstein modes in Critical Gravity. We have shown that the Critical Gravity action is quadratic in the Bach tensor, explaining the vanishing mass for the black holes and the gravitons, when constrained in the Einstein branch of the theory. Moreover, using the Noether-Wald method, we showed that the conserved charges are zero when evaluated for Einstein spacetimes, making this sector trivial. Thus, the only non-vanishing contribution is coming from non-Einstein spacetimes.

This formulation plays a crucial role when going to the computation of the holographic correlation functions in Critical Gravity, as it provides a shortcut in their derivation. Indeed, the new formulation indicates that the variation of the action is partially renormalized by the presence of the GB term, which is implicit in the Critical Gravity action. Switching on the logarithmic source at the boundary, additional counterterms depending on extrinsic curvature and its covariant derivatives were proposed. The new set of counterterms is not compatible with a Dirichlet boundary condition for h_{ij}. Nevertheless, the variation of the action is finite and the variational principle is well-posed by fixing the sources of the holographic duals at the boundary. Thus, the boundary conditions we impose are consistent with the holographic interpretation of the theory considering a relaxed AdS fall-off.

It is worth mentioning that mimicking the Topological Regularization scheme in the presence of a logarithmic source at the boundary is not a trivial issue, even if the presence of extrinsic counterterm would be a hint of a possible connection. The reason is that fine tuning the GB term considering the relaxed asymptotic behavior of the curvature tensor is highly non-trivial, as the source $b_{(0)ij}$ is neither a parameter nor a covariant field in the Lagrangian.

In the last section there were calculated the one- and two-point correlation functions of the boundary theory, recovering a structure that corresponds to a LCFT, as expected.

Appendix A

Completely symmetric traces and symmetric polynomials

We introduce the $d \times d$ matrices $\mathbb{A} = \left[A^i_j\right]$, the tensor $\mathbb{W} = [W^{ij}_{kl}]$ antisymmetric in the pairs of indices and the completely symmetric traces $\langle \cdots \rangle$ defined in terms of the generalized Kronecker delta $\delta^{j_1 \cdots j_p}_{i_1 \cdots i_p}$. The trace that acts on the tensorial product of the p $(0 \leq p \leq d)$ matrices. In this notation, we write

$$\begin{aligned} \delta^i_j &\rightarrow \mathbb{I}\,, \\ K^i_j &\rightarrow \mathbb{K}\,, \\ \mathcal{S}^i_j &\rightarrow \mathbb{S}\,, \\ W^{ij}_{kl} &\rightarrow \mathbb{W}\,, \\ \delta^{j_1 \cdots j_p}_{i_1 \cdots i_p} &\rightarrow \langle \cdots \rangle\,. \end{aligned} \tag{A.1}$$

In this notation, all tensors commute within the symmetric trace $\langle \cdots \rangle$. We also drop writing $\otimes$ in the tensorial product for the sake of simplicity (e.g., $\mathbb{A} \otimes \mathbb{B} \equiv \mathbb{A}\mathbb{B}$). Useful symmetric traces are

$$\begin{aligned} \left\langle \mathbb{I}^{2n-1} \right\rangle &= (2n-1)!\,, \\ \left\langle \mathbb{I}^{2n-2}\mathbb{A} \right\rangle &= (2n-2)!\,\langle \mathbb{A} \rangle\,, \\ \langle \mathbb{I}^p \mathbb{A} \rangle &= p!\,\langle \mathbb{A} \rangle\,, \\ \langle \mathbb{I}\mathbb{A}^p \rangle &= (d-p)\,\,\langle \mathbb{A}^p \rangle\,, \\ \left\langle \mathbb{I}^{d-p}\mathbb{A}^p \right\rangle &= (d-p)!\,\langle \mathbb{A}^p \rangle\,. \end{aligned} \tag{A.2}$$

Symmetric traces $\langle \cdots \rangle$ can be expanded in terms of the usual traces $\mathrm{Tr}(\cdots)$ which act on square matrices only, for example

$$\begin{aligned}
\langle 1 \rangle &= 1\,, \\
\langle \mathbb{A} \rangle &= \mathrm{Tr}\,\mathbb{A}\,, \\
\left\langle \mathbb{A}^2 \right\rangle &= (\mathrm{Tr}\,\mathbb{A})^2 - \mathrm{Tr}\,\mathbb{A}^2\,, \\
&\ \vdots \\
\left\langle \mathbb{A}^d \right\rangle &= d!\,\det \mathbb{A}\,,
\end{aligned} \tag{A.3}$$

or in case of tensorial product of matrices,

$$\begin{aligned}
\langle \mathbb{A} \rangle &= \mathrm{Tr}\,\mathbb{A}\,, \\
\langle \mathbb{A}\mathbb{B} \rangle &= \mathrm{Tr}\,\mathbb{A}\,\mathrm{Tr}\,\mathbb{B} - \mathrm{Tr}\,(\mathbb{A}\mathbb{B})\,, \\
\left\langle \mathbb{A}^2 \right\rangle &= (\mathrm{Tr}\,\mathbb{A})^2 - \mathrm{Tr}\mathbb{A}^2\,, \\
\langle \mathbb{A}\mathbb{B}\mathbb{C} \rangle &= \mathrm{Tr}\,\mathbb{A}\,\mathrm{Tr}\,\mathbb{B}\,\mathrm{Tr}\,\mathbb{C} + 2\mathrm{Tr}\,(\mathbb{A}\mathbb{B}\mathbb{C}) - \mathrm{Tr}\,\mathbb{A}\,\mathrm{Tr}\,(\mathbb{B}\mathbb{C}) - \mathrm{Tr}\,\mathbb{B}\,\mathrm{Tr}\,(\mathbb{A}\mathbb{C}) - \mathrm{Tr}\,\mathbb{C}\,\mathrm{Tr}\,(\mathbb{A}\mathbb{B})\,, \\
\left\langle \mathbb{A}^2\mathbb{B} \right\rangle &= (\mathrm{Tr}\,\mathbb{A})^2\,\mathrm{Tr}\,\mathbb{B} + 2\mathrm{Tr}\left(\mathbb{A}^2\mathbb{B}\right) - 2\mathrm{Tr}\,\mathbb{A}\,\mathrm{Tr}\,(\mathbb{A}\mathbb{B}) - \mathrm{Tr}\,\mathbb{A}^2\,\mathrm{Tr}\,\mathbb{B}\,, \\
\left\langle \mathbb{A}^3 \right\rangle &= (\mathrm{Tr}\,\mathbb{A})^3 + 2\mathrm{Tr}\,\mathbb{A}^3 - 3\mathrm{Tr}\,\mathbb{A}\,\mathrm{Tr}\,\mathbb{A}^2\,, \\
\left\langle \mathbb{A}^4 \right\rangle &= (\mathrm{Tr}\,\mathbb{A})^4 + 8\mathrm{Tr}\,\mathbb{A}\,\mathrm{Tr}\,\mathbb{A}^3 - 6\,(\mathrm{Tr}\,\mathbb{A})^2\,\mathrm{Tr}\,\mathbb{A}^2 + 3\left(\mathrm{Tr}\,\mathbb{A}^2\right)^2 - 6\mathrm{Tr}\,\mathbb{A}^4\,.
\end{aligned} \tag{A.4}$$

Note that the relation between the symmetric and standard traces is '$1-1$', so that we can also write the inverse of the previous relation,

$$\begin{aligned}
\mathrm{Tr}\,\mathbb{A} &= \langle \mathbb{A} \rangle\,, \\
\mathrm{Tr}\,(\mathbb{A}\mathbb{B}) &= \langle \mathbb{A} \rangle \langle \mathbb{B} \rangle - \langle \mathbb{A}\mathbb{B} \rangle\,, \\
\mathrm{Tr}\left(\mathbb{A}^2\right) &= \langle \mathbb{A} \rangle^2 - \left\langle \mathbb{A}^2 \right\rangle\,, \\
\mathrm{Tr}\,(\mathbb{A}\mathbb{B}\mathbb{C}) &= \langle \mathbb{A} \rangle \langle \mathbb{B} \rangle \langle \mathbb{C} \rangle + \frac{1}{2}\left(\langle \mathbb{A}\mathbb{B}\mathbb{C} \rangle - \langle \mathbb{A} \rangle \langle \mathbb{B}\mathbb{C} \rangle - \langle \mathbb{B} \rangle \langle \mathbb{A}\mathbb{C} \rangle - \langle \mathbb{C} \rangle \langle \mathbb{A}\mathbb{B} \rangle\right)\,, \\
\mathrm{Tr}\left(\mathbb{A}^2\mathbb{B}\right) &= \langle \mathbb{A} \rangle^2 \langle \mathbb{B} \rangle - \langle \mathbb{A} \rangle \langle \mathbb{A}\mathbb{B} \rangle + \frac{1}{2}\left(\left\langle \mathbb{A}^2\mathbb{B} \right\rangle - \left\langle \mathbb{A}^2 \right\rangle \langle \mathbb{B} \rangle\right)\,, \\
\mathrm{Tr}\left(\mathbb{A}^3\right) &= \langle \mathbb{A} \rangle^3 + \frac{1}{2}\left(\left\langle \mathbb{A}^3 \right\rangle - 3\,\langle \mathbb{A} \rangle \left\langle \mathbb{A}^2 \right\rangle\right)\,, \\
\mathrm{Tr}\left(\mathbb{A}^4\right) &= \langle \mathbb{A} \rangle^4 - \frac{1}{6}\left\langle \mathbb{A}^4 \right\rangle + \frac{2}{3}\langle \mathbb{A} \rangle \left\langle \mathbb{A}^3 \right\rangle - 2\,\langle \mathbb{A} \rangle^2 \left\langle \mathbb{A}^2 \right\rangle + \frac{1}{2}\left\langle \mathbb{A}^2 \right\rangle^2\,.
\end{aligned} \tag{A.5}$$

It is also useful to make a relation with the symmetric polynomials $P_p(\mathbb{A})$ of order p,

$$\begin{aligned} P_0(\mathbb{A}) &= 1\,, \\ P_1(\mathbb{A}) &= \mathrm{Tr}\,\mathbb{A}\,, \\ P_2(\mathbb{A}) &= \frac{1}{2}\left[(\mathrm{Tr}\,\mathbb{A})^2 - \mathrm{Tr}\,\mathbb{A}^2\right]\,, \\ &\vdots \\ P_d(\mathbb{A}) &= \det\mathbb{A}\,, \end{aligned} \tag{A.6}$$

which are conveniently defined by

$$\det\left(\mathbb{I} + u\,\mathbb{A}\right) = e^{\mathrm{Tr}\ln(\mathbb{I}+u\,\mathbb{A})} = \sum_{p=0}^{d} u^p P_p(\mathbb{A})\,. \tag{A.7}$$

A relation between the polynomials and the traces is

$$P_p(\mathbb{A}) = \frac{1}{p!}\,\langle\mathbb{A}^p\rangle = \frac{1}{(d-p)!p!}\,\left\langle\mathbb{A}^p\mathbb{I}^{d-p}\right\rangle\,. \tag{A.8}$$

Appendix B

Compilation of useful identities

The sums that appear in the counterterms in even dimensions have the form

$$\Xi_p = \sum_{k=1}^{n-1} \frac{(-4)^k \, k!}{(n-1-k)! \, (2k+1-p)!} \, . \tag{B.1}$$

The can be expressed in terms of the hypergeometric function. First few values are

$$\begin{aligned}
\Xi_0 &= -\frac{2}{(2n-1)\,(n-2)!}\,, \\
\Xi_1 &= -\frac{2}{(2n-3)\,(n-2)!}\,, \\
\Xi_2 &= \frac{4}{(2n-3)\,(2n-5)\,(n-2)!}\,, \\
\Xi_3 &= -\frac{12}{(2n-5)\,(2n-7)\,(n-2)!}\,, \\
\Xi_4 &= \frac{96}{(2n-5)\,(2n-7)\,(2n-9)\,(n-3)!}\,.
\end{aligned} \tag{B.2}$$

We also need the sum

$$\sum_{k=1}^{n-1} \frac{(n-1-k)!^2}{2^{2k+1}(2n-2k-1)!} = -\frac{1}{2^{2n-1}} + \frac{(n-1)\,(n-1)!^2}{4\,(2n-3)!}\,. \tag{B.3}$$

Useful integral used in the text is

$$\int_0^1 du \, \left(1-u^2\right)^k = \frac{2^{2k} k!^2}{(2k+1)!}\,. \tag{B.4}$$

In odd dimensions, the following coefficients are also useful,

$$\begin{aligned} a_k &= \frac{k!\,(n-1-k)!^2}{n!(n-1)!2^{2k+1}(2n-2k-1)!}\,, \\ b_{kl} &= \frac{(-1)^l\,(2l+1)!}{l!\,(k-l)!}\,, \end{aligned} \tag{B.5}$$

whose first few values are

$$\begin{aligned} a_0 &= \frac{1}{2n(2n-1)!}\,, \\ a_1 &= \frac{1}{2n(2n-2)(2n-2)!}\,, \\ a_2 &= \frac{1}{n\,(2n-2)^2\,(2n-4)^2\,(2n-5)!}\,. \end{aligned} \tag{B.6}$$

Defining the sums

$$\Theta_{km_0} = \sum_{l=0}^{k} \frac{b_{kl}}{(2l+1-m_0)!}\,, \qquad k \geq 1\,. \tag{B.7}$$

we have, in particular,

$$\begin{aligned} \Theta_{k1} &= -2\delta_{k1}\,, \\ \Theta_{k2} &= -6\delta_{k1} + 4\delta_{k2}\,, \end{aligned} \tag{B.8}$$

and only Θ_{km_0} ($k \leq m_0$) are non-vanishing because $\Theta_{km_0} = 0$ ($k > m_0$).

Appendix C

Lowest-order approximation of the binomial expansion

The following notation is useful

$$K^{j_1}_{(n)i_1}\dots K^{j_p}_{(n)i_p}=\left(K^{j_1}_{(n)i_1}\right)^p\,, \tag{C.1}$$

for the product of the coefficients of the extrinsic curvature.

In order to simplify the formulas of Chapter 2, we introduce a series of auxiliary variables

$$\begin{aligned}
M_0 &= \left(K^{j_2}_{(0)i_2}\right)^{2n-2}, M_{2,0}=K^{j_2}_{(2)i_2}\left(K^{j_3}_{(0)i_3}\right)^{2n-3},\\
M_{4,0} &= K^{j_2}_{(4)i_2}\left(K^{j_3}_{(0)i_3}\right)^{2n-3}, M_{2,2}=\left(K^{j_2}_{(2)i_2}\right)^{2}\left(K^{j_3}_{(0)i_3}\right)^{2n-4},\\
M_{6,0} &= K^{j_2}_{(6)i_2}\left(K^{j_3}_{(0)i_3}\right)^{2n-3}, M_{2,2,2}=\left(K^{j_2}_{(2)i_2}\right)^{3}\left(K^{j_3}_{(0)i_3}\right)^{2n-5},\\
M_{2,4} &= K^{j_2}_{(2)i_2}K^{j_3}_{(4)i_3}\left(K^{j_3}_{(0)i_3}\right)^{2n-4}.
\end{aligned}$$

In even-dimensional counterterms we define

$$\begin{aligned}
\Pi_2 &= 2\left(n-1\right)t^{2n-4}\left(1-t^2\right)M_{2,0}\,,\\
\Pi_4 &= \left(n-1\right)t^{2n-6}\left(1-t^2\right)\left[2t^2M_{4,0}-\left(2\left(n-2\right)-\left(2n-3\right)t^2\right)M_{2,2}\right],\\
\Pi_6 &= 2\left(n-1\right)t^{2n-8}\left(1-t^2\right)\left[t^4M_{6,0}+\left(n-2\right)\left(n-1\right)\left(\frac{2n}{3}\left(1-t^2\right)-1\right)M_{2,2,2}+\right.\\
&+ \left.t^2\left(\left(2n-5\right)t^2-2\left(n-2\right)\right)M_{2,4}\right],
\end{aligned}$$

while in the odd dimensions, we use

$$
\begin{aligned}
\Sigma_2 &= 2\,(n-1)\left(1-t^2\right)\left(s^2-t^2\right)^{n-2} M_{2,0}\,, \\
\Sigma_4 &= (n-1)\left(1-t^2\right)\left(s^2-t^2\right)^{n-3}\left[2\left(s^2-t^2\right)M_{4,0}+\left(s^2-(2n-3)\,t^2+2\,(n-2)\right)M_{2,2}\right]\,, \\
\Sigma_6 &= 2\,(n-1)\left(1-t^2\right)\left(s^2-t^2\right)^{n-4}\left[\left(s^2-t^2\right)M_{6,0}+\left(s^2-t^2\right)\left(s^2-(2n-3)\,t^2+2\,(n-2)\right)\right. \\
&+ \left.(n-2)\left(1-t^2\right)\left(s^2-(2n-3)\,t^2+\frac{2}{3}\,(n-3)\right)M_{2,2,2}\right]\,,
\end{aligned}
$$

in order to shorten the expressions. Moreover, we replace the product of the $(n-1)$ parentheses, with the expression $\left[\frac{1}{2}W^{j_2j_3}_{i_2i_3}+\left(1-t^2\right)K^{j_2}_{i_2}K^{j_3}_{i_3}-\frac{1}{\ell^2}\delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\right]^{n-1}$
.

Thus, the lowest-order approximation of the binomial expansion in even-dimensions gives

$$
\begin{aligned}
&\Big[-t^2\left(K^{j_2}_{(0)i_2}\right)^2+2\rho\left(1-t^2\right)K^{j_2}_{(0)i_2}K^{j_3}_{(2)i_3}+\rho^2\left(1-t^2\right)\left(\left(K^{j_2}_{(2)i_2}\right)^2+2K^{j_2}_{(0)i_2}K^{j_3}_{(4)i_3}\right) \\
&+ 2\rho^3\left(1-t^2\right)\left(K^{j_2}_{(0)i_2}K^{j_3}_{(6)i_3}+K^{j_2}_{(2)i_2}K^{j_3}_{(4)i_3}\right)\Big]^{n-1} = (-1)^n\left(-t^{2n-2}M_0+\rho\Pi_2+\rho^2\Pi_4+\rho^3\Pi_6\right)\,.
\end{aligned}
$$

When applied in the odd-dimensional case we get

$$
\begin{aligned}
&\Big[\left(s^2-t^2\right)K^{j_2}_{(0)i_2}K^{j_3}_{(0)i_3}+2\rho\left(1-t^2\right)K^{j_2}_{(0)i_2}K^{j_3}_{(2)i_3}+\rho^2\left(1-t^2\right)\left(K^{j_2}_{(2)i_2}K^{j_3}_{(2)i_3}+2K^{j_2}_{(0)i_2}K^{j_3}_{(4)i_3}\right) \\
&+ 2\rho^3\left(1-t^2\right)\left(K^{j_2}_{(2)i_2}K^{j_3}_{(4)i_3}+K^{j_2}_{(0)i_2}K^{j_3}_{(6)i_3}\right)\Big]^{n-1} = \left(s^2-t^2\right)^{n-1}M_0+\rho\Sigma_2+\rho^2\Sigma_4+\rho^3\Sigma_6\,.
\end{aligned}
$$

Appendix D

Parametric Integrations

The following integral is proved useful for the calculation of the parametric integrations that follow

$$\int_0^1 du \left(1-u^2\right)^k = \frac{2^{2k}k!^2}{(2k+1)!} \Leftrightarrow \tag{D.1}$$

$$\Leftrightarrow \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^{k-i}}{2i+1} = (-1)^{-k} \frac{2^{2k}k!^2}{(2k+1)!}\,. \tag{D.2}$$

Based on this, we derive the parametric integrations below that are used in the derivation of the odd-dimensional Kounterterms.

$$\begin{aligned}\int_0^1 dt \int_0^t ds \left(s^2-t^2\right)^{n-1} &= \int_0^1 dt \int_0^t ds \sum_{i=0}^{n-1} \binom{n-1}{i} s^{2i} \left(-1\right)^{n-1-i} t^{2(n-1-i)} \\ &= \frac{(-1)^{-n+1}\, 2^{2n-2} \left(n-1\right)!^2}{(2n)!}\,,\end{aligned}$$

$$\int_0^1 dt \int_0^t ds\, 2\left(n-1\right)\left(1-t^2\right)\left(s^2-t^2\right)^{n-2} = \frac{(-1)^{-n+2}}{n} \frac{2^{2n-4}\left(n-2\right)!^2}{(2n-3)!}\,,$$

$$\int_0^1 dt \int_0^t ds \left(1-t^2\right)^2 \left(s^2-t^2\right)^{n-3} = \frac{\left(-1\right)^{-n+3} 2^{2n-6} (n-3)!^2}{n\,(n-1)\,(n-2)\,(2n-5)!},$$

$$\int_0^1 dt \int_0^t ds \left(2-3t^2\right) \left(s^2-t^2\right)^{n-2} = \frac{\left(-1\right)^{-n+1} 2^{2n-4} (n-3)\,(n-2)!^2}{n\,(2n-2)!},$$

$$\frac{1}{4}\int_0^1 dt \int_0^t ds \left(s^2-t^2\right)^{n-1} = \frac{\left(-1\right)^{-n+1} 2^{2n-4} (n-1)!^2}{(2n)!},$$

$$\int_0^1 dt \int_0^t ds \left(s^2-t^2\right)^{n-2} t^2 = \frac{\left(-1\right)^{-n+2} 2^{2n-4} (n-2)!^2}{2n\,(2n-3)!},$$

$$\int_0^1 dt \int_0^t ds \left(1-t^2\right) \left(s^2-t^2\right)^{n-3} = \frac{\left(-1\right)^{-n+3} 2^{2n-6} (n-3)!^2}{(n-1)\,(2n-4)!},$$

$$\int_0^1 dt \int_0^t ds \left(1-t^2\right)^3 \left(s^2-t^2\right)^{n-4} = \frac{\left(-1\right)^{-n+4} 3}{n\,(n-1)\,(n-2)\,(n-3)} \frac{2^{2n-8} (n-4)!^2}{(2n-7)!},$$

$$\int_0^1 dt \int_0^t ds \left(1-t^2\right) t^2 \left(s^2-t^2\right)^{n-3} = \frac{\left(-1\right)^{-n+3} 2^{2n-7} (n-3)!^2}{n\,(n-1)\,(2n-5)!}.$$

Bibliography

[1] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity", *Int. J. Theor. Phys.*, vol. 38, pp. 1113–1133, 1999, [Adv. Theor. Math. Phys.2,231(1998)]. DOI: 10.1023/A:1026654312961,10.4310/ATMP.1998.v2.n2.a1. arXiv: hep-th/9711200 [hep-th].

[2] E. Witten, "Anti-de Sitter space and holography", *Adv. Theor. Math. Phys.*, vol. 2, pp. 253–291, 1998. arXiv: hep-th/9802150 [hep-th].

[3] C. Fefferman and C. Robin Graham, "Conformal Invariants", *in Elie Cartan et les Math´ematiques d'aujourd'hui (Ast´erisque, 1985)*, vol. 95, 1985.

[4] S. de Haro, S. N. Solodukhin, and K. Skenderis, "Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence", *Commun. Math. Phys.*, vol. 217, pp. 595–622, 2001. DOI: 10.1007/s002200100381. arXiv: hep-th/0002230 [hep-th].

[5] M. Henningson and K. Skenderis, "Holography and the Weyl anomaly", *Fortsch. Phys.*, vol. 48, pp. 125–128, 2000. DOI: 10.1002/(SICI)1521-3978(20001)48:1/3<125::AID-PROP125>3.0.CO;2-B,10.1002/(SICI)1521-3978(20001)48:1/3<125::AID-PROP125>3.3.CO;2-2. arXiv: hep-th/9812032 [hep-th].

[6] M. Henningson and K. Skenderis, "The Holographic Weyl anomaly", *JHEP*, vol. 07, p. 023, 1998. DOI: 10.1088/1126-6708/1998/07/023. arXiv: hep-th/9806087 [hep-th].

[7] R. Olea, "Mass, angular momentum and thermodynamics in four-dimensional Kerr-AdS black holes", *JHEP*, vol. 06, p. 023, 2005. DOI: 10.1088/1126-6708/2005/06/023. arXiv: hep-th/0504233 [hep-th].

[8] ——, "Regularization of odd-dimensional AdS gravity: Kounterterms", *JHEP*, vol. 04, p. 073, 2007. DOI: 10.1088/1126-6708/2007/04/073. arXiv: hep-th/0610230 [hep-th].

[9] G. Anastasiou and R. Olea, "From conformal to Einstein Gravity", *Phys. Rev.*, vol. D94, no. 8, p. 086 008, 2016. DOI: 10.1103/PhysRevD.94.086008. arXiv: 1608.07826 [hep-th].

[10] G. Anastasiou, R. Olea, and D. Rivera-Betancour, "Noether-Wald energy in Critical Gravity", 2017. arXiv: 1707.00341 [hep-th].

[11] G. Anastasiou and R. Olea, "Holographic correlation functions in Critical Gravity", *JHEP*, vol. 11, p. 019, 2017. DOI: 10.1007/JHEP11(2017)019. arXiv: 1709.01174 [hep-th].

[12] A. Ashtekar and A. Magnon, "Asymptotically anti-de Sitter space-times", *Class. Quant. Grav.*, vol. 1, pp. L39–L44, 1984. DOI: 10.1088/0264-9381/1/4/002.

[13] M. Henneaux and C. Teitelboim, "Asymptotically anti-De Sitter Spaces", *Commun. Math. Phys.*, vol. 98, pp. 391–424, 1985. DOI: 10.1007/BF01205790.

[14] O. Miskovic and R. Olea, "Background-independent charges in Topologically Massive Gravity", *JHEP*, vol. 12, p. 046, 2009. DOI: 10.1088/1126-6708/2009/12/046. arXiv: 0909.2275 [hep-th].

[15] P. Kraus, F. Larsen, and R. Siebelink, "The gravitational action in asymptotically AdS and flat space-times", *Nucl. Phys.*, vol. B563, pp. 259–278, 1999. DOI: 10.1016/S0550-3213(99)00549-0. arXiv: hep-th/9906127 [hep-th].

[16] I. Papadimitriou and K. Skenderis, "AdS / CFT correspondence and geometry", *IRMA Lect. Math. Theor. Phys.*, vol. 8, pp. 73–101, 2005. DOI: 10.4171/013-1/4. arXiv: hep-th/0404176 [hep-th].

[17] T. Eguchi, P. B. Gilkey, and A. J. Hanson, "Gravitation, Gauge Theories and Differential Geometry", *Phys. Rept.*, vol. 66, p. 213, 1980. DOI: 10.1016/0370-1573(80)90130-1.

[18] P. Mora, R. Olea, R. Troncoso, and J. Zanelli, "Transgression forms and extensions of Chern-Simons gauge theories", *JHEP*, vol. 02, p. 067, 2006. DOI: 10.1088/1126-6708/2006/02/067. arXiv: hep-th/0601081 [hep-th].

[19] M. H. Goroff and A. Sagnotti, "The Ultraviolet Behavior of Einstein Gravity", *Nucl. Phys.*, vol. B266, pp. 709–736, 1986. DOI: 10.1016/0550-3213(86)90193-8.

[20] K. S. Stelle, "Renormalization of Higher Derivative Quantum Gravity", *Phys. Rev.*, vol. D16, pp. 953–969, 1977. DOI: 10.1103/PhysRevD.16.953.

[21] D. M. Capper and M. J. Duff, "Conformal Anomalies and the Renormalizability Problem in Quantum Gravity", *Phys. Lett.*, vol. A53, p. 361, 1975. DOI: 10.1016/0375-9601(75)90030-4.

[22] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, "Massive Gravity in Three Dimensions", *Phys. Rev. Lett.*, vol. 102, p. 201 301, 2009. DOI: 10.1103/PhysRevLett.102.201301. arXiv: 0901.1766 [hep-th].

[23] S. Deser and B. Tekin, "Massive, topologically massive, models", *Class. Quant. Grav.*, vol. 19, pp. L97–L100, 2002. DOI: 10.1088/0264-9381/19/11/101. arXiv: hep-th/0203273 [hep-th].

[24] H. Lu and C. N. Pope, "Critical Gravity in Four Dimensions", *Phys. Rev. Lett.*, vol. 106, p. 181 302, 2011. DOI: 10.1103/PhysRevLett.106.181302. arXiv: 1101.1971 [hep-th].

[25] S. L. Adler, "Einstein Gravity as a Symmetry Breaking Effect in Quantum Field Theory", *Rev. Mod. Phys.*, vol. 54, p. 729, 1982, [Erratum: Rev. Mod. Phys.55,837(1983)]. DOI: 10.1103/RevModPhys.54.729.

[26] P. D. Mannheim and D. Kazanas, "Exact Vacuum Solution to Conformal Weyl Gravity and Galactic Rotation Curves", *Astrophys. J.*, vol. 342, pp. 635–638, 1989. DOI: 10.1086/167623.

[27] P. D. Mannheim, "Alternatives to dark matter and dark energy", *Prog. Part. Nucl. Phys.*, vol. 56, pp. 340–445, 2006. DOI: 10.1016/j.ppnp.2005.08.001. arXiv: astro-ph/0505266 [astro-ph].

[28] ——, "Making the Case for Conformal Gravity", *Found. Phys.*, vol. 42, pp. 388–420, 2012. DOI: 10.1007/s10701-011-9608-6. arXiv: 1101.2186 [hep-th].

[29] P. D. Mannheim and J. G. O'Brien, "Impact of a global quadratic potential on galactic rotation curves", *Phys. Rev. Lett.*, vol. 106, p. 121 101, 2011. DOI: 10.1103/PhysRevLett.106.121101. arXiv: 1007.0970 [astro-ph.CO].

[30] N. Berkovits and E. Witten, "Conformal supergravity in twistor-string theory", *JHEP*, vol. 08, p. 009, 2004. DOI: 10.1088/1126-6708/2004/08/009. arXiv: hep-th/0406051 [hep-th].

[31] J. Maldacena, "Einstein Gravity from Conformal Gravity", 2011. arXiv: 1105.5632 [hep-th].

[32] O. Miskovic and R. Olea, "Topological regularization and self-duality in four-dimensional anti-de Sitter gravity", *Phys. Rev.*, vol. D79, p. 124 020, 2009. DOI: 10.1103/PhysRevD.79.124020. arXiv: 0902.2082 [hep-th].

[33] C. M. Bender and P. D. Mannheim, "No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck oscillator model", *Phys. Rev. Lett.*, vol. 100, p. 110 402, 2008. DOI: 10.1103/PhysRevLett.100.110402. arXiv: 0706.0207 [hep-th].

[34] S. Deser and R. I. Nepomechie, "Gauge Invariance Versus Masslessness in De Sitter Space", *Annals Phys.*, vol. 154, p. 396, 1984. DOI: 10.1016/0003-4916(84)90156-8.

[35] S. Deser and A. Waldron, "Gauge invariances and phases of massive higher spins in (A)dS", *Phys. Rev. Lett.*, vol. 87, p. 031 601, 2001. DOI: 10.1103/PhysRevLett.87.031601. arXiv: hep-th/0102166 [hep-th].

[36] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity", *Commun. Math. Phys.*, vol. 208, pp. 413–428, 1999. DOI: 10.1007/s002200050764. arXiv: hep-th/9902121 [hep-th].

[37] M. T. Anderson, "Geometric aspects of the AdS / CFT correspondence", *IRMA Lect. Math. Theor. Phys.*, vol. 8, pp. 1–31, 2005. arXiv: hep-th/0403087 [hep-th].

[38] K. S. Stelle, "Classical Gravity with Higher Derivatives", *Gen. Rel. Grav.*, vol. 9, pp. 353–371, 1978. DOI: 10.1007/BF00760427.

[39] H. Lu, Y. Pang, and C. N. Pope, "Conformal Gravity and Extensions of Critical Gravity", *Phys. Rev.*, vol. D84, p. 064 001, 2011. DOI: 10.1103/PhysRevD.84.064001. arXiv: 1106.4657 [hep-th].

[40] M. Alishahiha and R. Fareghbal, "D-Dimensional Log Gravity", *Phys. Rev.*, vol. D83, p. 084 052, 2011. DOI: 10.1103/PhysRevD.83.084052. arXiv: 1101.5891 [hep-th].

[41] E. A. Bergshoeff, O. Hohm, J. Rosseel, and P. K. Townsend, "Modes of Log Gravity", *Phys. Rev.*, vol. D83, p. 104 038, 2011. DOI: 10.1103/PhysRevD.83.104038. arXiv: 1102.4091 [hep-th].

[42] I. Gullu, M. Gurses, T. C. Sisman, and B. Tekin, "AdS Waves as Exact Solutions to Quadratic Gravity", *Phys. Rev.*, vol. D83, p. 084 015, 2011. DOI: 10.1103/PhysRevD.83.084015. arXiv: 1102.1921 [hep-th].

[43] V. Gurarie, "Logarithmic operators in conformal field theory", *Nucl. Phys.*, vol. B410, pp. 535–549, 1993. DOI: 10.1016/0550-3213(93)90528-W. arXiv: hep-th/9303160 [hep-th].

[44] J. S. Caux, I. I. Kogan, and A. M. Tsvelik, "Logarithmic operators and hidden continuous symmetry in critical disordered models", *Nucl. Phys.*, vol. B466, pp. 444–462, 1996, [,2223(1995)]. DOI: 10.1016/0550-3213(96)00118-6. arXiv: hep-th/9511134 [hep-th].

[45] O. Miskovic, R. Olea, and M. Tsoukalas, "Renormalized AdS action and Critical Gravity", *JHEP*, vol. 08, p. 108, 2014. DOI: 10.1007/JHEP08(2014)108. arXiv: 1404.5993 [hep-th].

[46] S. Deser and B. Tekin, "Energy in generic higher curvature gravity theories", *Phys. Rev. D*, vol. **67**, p. 084 009, 2003. arXiv: hep-th/0212292 [hep-th].

[47] S. Deser and B. Tekin, "Gravitational energy in quadratic curvature gravities", *Phys. Rev. Lett.*, vol. 89, p. 101 101, 2002. DOI: 10.1103/PhysRevLett.89.101101. arXiv: hep-th/0205318 [hep-th].

[48] V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy", *Phys. Rev. D*, vol. **50**, p. 846, 1994. arXiv: gr-qc/9403028 [gr-qc].

[49] V. Iyer and R. M. Wald, "A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes", *Phys. Rev.*, vol. D52, pp. 4430–4439, 1995. DOI: 10.1103/PhysRevD.52.4430. arXiv: gr-qc/9503052 [gr-qc].

[50] S. W. MacDowell and F. Mansouri, “Unified Geometric Theory of Gravity and Supergravity”, *Phys. Rev. Lett.*, vol. 38, p. 739, 1977, [Erratum: Phys. Rev. Lett.38,1376(1977)]. DOI: 10.1103/PhysRevLett.38.1376, 10.1103/PhysRevLett.38.739.

[51] M. Porrati and M. M. Roberts, “Ghosts of Critical Gravity”, *Phys. Rev.*, vol. D84, p. 024 013, 2011. DOI: 10.1103/PhysRevD.84.024013. arXiv: 1104.0674 [hep-th].

[52] D. Grumiller and N. Johansson, “Consistent boundary conditions for cosmological topologically massive gravity at the chiral point”, *Int. J. Mod. Phys.*, vol. D17, pp. 2367–2372, 2009. DOI: 10.1142/S0218271808014096. arXiv: 0808.2575 [hep-th].

[53] N. Johansson, A. Naseh, and T. Zojer, “Holographic two-point functions for 4d log-gravity”, *JHEP*, vol. 09, p. 114, 2012. DOI: 10.1007/JHEP09(2012)114. arXiv: 1205.5804 [hep-th].

[54] M. Alishahiha and A. Naseh, “Holographic renormalization of new massive gravity”, *Phys. Rev.*, vol. D82, p. 104 043, 2010. DOI: 10.1103/PhysRevD.82.104043. arXiv: 1005.1544 [hep-th].

[55] K. Skenderis, M. Taylor, and B. C. van Rees, “Topologically Massive Gravity and the AdS/CFT Correspondence”, *JHEP*, vol. 09, p. 045, 2009. DOI: 10.1088/1126-6708/2009/09/045. arXiv: 0906.4926 [hep-th].

[56] L. Rozansky and H. Saleur, “Quantum field theory for the multivariable Alexander-Conway polynomial”, *Nucl. Phys.*, vol. B376, pp. 461–509, 1992. DOI: 10.1016/0550-3213(92)90118-U.

[57] H. Saleur, “Polymers and percolation in two-dimensions and twisted N=2 supersymmetry”, *Nucl. Phys.*, vol. B382, pp. 486–531, 1992. DOI: 10.1016/0550-3213(92)90657-W. arXiv: hep-th/9111007 [hep-th].

[58] V. Gurarie and A. W. W. Ludwig, “Conformal algebras of 2-D disordered systems”, *J. Phys.*, vol. A35, pp. L377–L384, 2002. DOI: 10.1088/0305-4470/35/27/101. arXiv: cond-mat/9911392 [cond-mat.dis-nn].

[59] M. R. Rahimi Tabar and S. Rouhani, “Turbulent two-dimensional magnetohydrodynamics and conformal field theory”, *Annals Phys.*, vol. 246, pp. 446–458, 1996. DOI: 10.1006/aphy.1996.0033. arXiv: hep-th/9503005 [hep-th].

[60] I. I. Kogan and A. M. Tsvelik, “Logarithmic operators in the theory of plateau transition”, *Mod. Phys. Lett.*, vol. A15, pp. 931–938, 2000. DOI: 10.1142/S0217732300000931, 10.1016/S0217-7323(00)00093-1. arXiv: hep-th/9912143 [hep-th].

[61] I. I. Kogan and N. E. Mavromatos, “World sheet logarithmic operators and target space symmetries in string theory”, *Phys. Lett.*, vol. B375, pp. 111–120, 1996. DOI: 10.1016/0370-2693(96)00195-5. arXiv: hep-th/9512210 [hep-th].

[62] V. Gurarie, "Logarithmic operators and logarithmic conformal field theories", *J. Phys.*, vol. A46, p. 494 003, 2013. DOI: 10.1088/1751-8113/46/49/494003. arXiv: 1303.1113 [cond-mat.stat-mech].