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First-order Lagrangian of Lovelock gravity and applications

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Abstract

We analytically explore spatial (normal) evolution aspects of Lovelock gravity. We show that the addition of Myers' terms to the action eliminates second-order normal derivatives. The connection between the Dirichlet problem and a first-order Lagrangian is then established. Then, we exhibit how the (spatial) Hamiltonian density can be directly computed with the Legendre transformation of that *first-order* Lagrangian. We use those results to analyse the behaviour of massive thin shells in the context of Lovelock gravity. In order to do so, we work out the variational principle, written in an adapted set of coordinates valid near the shell. We conclude that the quantity that must jump is the canonical momentum. Then, we calculate explicitly some expressions for spherically symmetric configurations.

Chapter 1

Introduction

Gravity, it is not an exaggeration to say that a significant part of civilization have been dedicated to study or defy it. Skyscrapers, rockets, planes, javelin throw, high jump, climbing, roller coasters, there is an infinitude of objects and situations. However, the concept of gravity has been elusive and many scientists contributed to the nowadays concept[1].

Between the 400 B.C. and 300 B.C., Plato's school grew around the idea of the five elements. Plato and his colleagues believed that all objects were composed of earth, water, air and fire. The fifth element, which they called *quintessence*, played the role of stuff from heavens, a godly element where all things are embroidered. Aristotle was part of the school. While Plato believed that all objects are a deviation from an ideal shape and they lack individuality, Aristotle proposed that each object is an absolute with potential or purpose. He divided the earthly elements, earth, water, air and fire, between its purposes. For him, earth and water were a source of gravity, while air and fire were a source of *levity* –not in the sense of lightness, but more like buoyancy–, a force completely opposed to gravity. Earth was the strongest source of gravity and fire was the strongest source of levity. Then, his concept of gravitation was an effect resulting from the different ratios of the four elements in all objects. Aristotle's interpretation of the universe was a sphere of earth at the center,

[1] The next paragraphs are based on the enjoyable story in Ref.[1]

layered by water, air and fire respectively. The fifth element, which he started calling *ether*, was the outermost layer where planets and stars live. Due to earth being the main source of gravity, object fall directly to its center. This geocentric view was reinforced by the belief of Earth being the center of the universe.

Aristotle's ideas had many flaws. In particular, he taught that there is a slight acceleration due to the proximity with the earth element, however, it was not enough to explain reality. For Aristotle, objects achieved the maximum constant descending velocity little after falling. In the 200 B.C., Strato doubted some of Aristotle's claims. He denied the existence of levity and attributed the displacement to the weight of objects. He also noticed that height played a role in the land impact of falling objects, in contrast with the idea of instant terminal velocity. In fact, Aristotle described projectile trajectory as two straight lines. Many years passed until this changed with the appearance of gunpowder around the 10th century in the East (Asia) and 13th century in the West (Europe). With the invention of cannons and other types of artillery guns, the marksmen noticed that the route of projectiles is curved. War was an important part of this epoch and there was special attention in this detail. Of course this was noticed by many people before, like for example archers. However, philosophers had status and never allowed their wisdom to be put to test. Thus, the interest in war played an important role in putting away pretenses and explore the truth. It was Tartaglia in the 16th century, the one to study in detail bullet trajectories. He developed a graduated tool for firing assistance and inquired the effect of the shooting angle. He was the first to document and affirm that Aristotle's idea of projectile route is wrong and that it pictures a curve along all the flight. However, it lacked definiteness. Later, in the 18th century, Robins strengthened the artillery theory studying the final velocity of bullets using its ballistic pendulum to measure the effect of air.

Even through many doubts arose in regard to Aristotle's postulates, it was not until the 17th century that someone with enough weight tackled gravity. At that time, there were not high speed cameras or chronometers to measure a falling process, which happens in an instant. Galileo Galilei devised a solution employing inclined planes to slow down the experiments. Also, he used water clocks to measure time for more accuracy. After a considerable amount of measurements, he realized that the gravitational acceleration is constant and the distance traveled from rest proportional to the square of the time employed descending. Also, he noticed that in absence of friction, the motion should be perpetual. At the end, after compiling an appropriate amount of evidence, Galileo found the shape of the trajectory of a projectile when the air influence is negligible: a parabola, bringing mathematics to the world of gravity for the first time.

Galileo was not the first, but the most decisive in abolish geocentrism. He noticed that the moon and sun were not perfect and that celestial bodies move. These observations, that he published, brought many scientists together, but also, a deep enmity from the church that regarded celestial bodies as perfect objects of god. Backed up by with Copernicus ideas, he devised a model where the Sun was the center: heliocentrism. Copernicos proposal had flaws, like the circular orbits he predicted, that were not accurate enough to describe the planetary movements. It was still restricted by the idea of ether, hampering the motion of planets. The pass of Halley comet changed this concept. Tycho, an enthusiast astronomer that studied and documented planetary motions noticed the eccentricity of Halley's path. In his last years, he joined hands with Kepler, who after his death inherited his voluminous records. With Tycho's data, Kepler observed that the orbits are elliptical, with the sun at one of the focal points. He developed his three laws of planetary motion, a huge bomb for the Church that professed the perfection of the heavens. However, even though Kepler understood the rules of planetary movements, he did not know why they moved.

The change came with the iconic Newton's apple. The story says that Newton was watching the moon when he saw a apple falling from a tree. This event inspired Newton to extrapolate the concept of gravity to the moon. He asked himself why the moon is not falling to Earth like the apple. He stipulated his three laws of motion.

He devised an early stage of calculus and used it to fathom gravity. He discovered the formula describing centrifugal force. He realized that gravity must be balancing the moon's centrifugal force. He had the insight to detect that without a force pulling the moon toward Earth, it would escape from our planet. He derived from his and Kepler's formulas that the constraining force must be proportional to the inverse of the orbital radius squared. Unfortunately, the data he needed to corroborate his formula lacked accuracy and he dropped the subject for a decade. Hook revitalized Newton writing to him to discuss planetary movement. Hook talked about central forces, that was exactly what Newton needed to solve the holes in his conclusions. Newton neglected Hook and started to study alone. First, he linked Kepler laws with central force calculations. With his newly-born calculus, he computed the planetary orbits using infinitesimal displacements and arrived to an elliptic orbit. Urged by Halley, Newton wrote the proofs and calculations and published them in a book divided in three tomes, *Principia*. His book built the basics for a new vision of motion and mechanics, the future core of classical physics. It was also the first time the concept of *inertia* appeared along with his law of universal gravitation $F = GMm/r^2$, where F is the gravitational force, M and m the masses of the sample objects, r their separation distance and G a constant, called Newton's or universal gravitational constant. The idea of force without contact was revolutionary for the time. Comet Halley observations played an important role. Scientist noticed that the comet in the sky in the late 17th century was the same saw roughly 70 years ago. Clairaut computed its orbit and predicted the next appearance in Earth's sky using Newtonian mechanics. Later, it was corroborated in the middles of the 18th century strengthening the credibility.

The physics of *Principia* gave scientists tools to predict the presence of more planets in the solar system. Their mutual interaction produced slight deviations from the perfect elliptical path. Among them, in the late 19th century, Mercury was special. The motion of Mercury in the segment nearest to the Sun, the perihelion, presented such deviations. Leverrier was convinced that there must be a planet waiting to be discovered, close enough to the Sun to avoid being detected before. A huge commotion followed the idea, with scientist all over the globe looking for the missing planet, which they named Vulcan. Unfortunately, as we already know in actuality, Vulkan was never found.

In the late 19th century, scientists started to ask themselves how do massive objects know that they are close to each other without any contact. They spoke of two great forces of nature: gravity and electromagnetism. Both obey the inverse square law and act across distance. Maxwell resumed all electric and magnetic behaviors in four equations. He notice that moving a magnet gives rise to waves that travel at the speed of light. Based on the understanding of mechanical waves, Maxwell believed that there is not such a thing as vacuum, but a mysterious substance, the ether. A weightless element that lacks any resistance to objects passing by. However, gravity was different, there is no repulsion between bodies, like in the case of opposite charges, only attraction. He attributed gravity to a local decrease of ether and never went beyond this idea.

The ether was in the minds of scientist in their attempt to improve the comprehension of gravity. In Newton's formulation of physics, an absolute rest frame existed. Then, if electromagnetic waves travel along ether, then there must be possible to measure a frame in rest with the ether. Maxwell had the idea of using celestial bodies to measure the changes in the velocity of light due to ether fluctuations. However, all efforts failed. It was hard to understand how the flux of a substance like ether was not affecting the light rays. Later, Fitzgerald made an unusual claim: the length of materials changes according to the square of the velocity. Surprisingly, Lorentz made the same remark in parallel, and the change would later be called Lorentz transformation. Lorentz noticed that the transformation named after him have full grasp of Maxwell's equations for moving bodies. At this point, physicists acknowledged the need to reinterpret space and time. At the start of the 20th century, people was already doubting the existence of ether and Poincaré appointed the need of new mechanics with the speed of light as an impassable limit.

Einstein, from young age, asked himself about the structure of time, space and light. He liked though experiments and employed them thoroughly to stimulate his intuition. He blindly believed in the speed of light as an insurmountable and absolute quantity, a cap. He though about the example of relative motion [2]. If you move inside a train, it is normal that an outer observer will see you moving at a speed that is the sum of the speed of the subject in the train frame plus the speed of the train. Then, light must be free of such notion, otherwise it would be easy to surpass the speed of light. However, this is a difficulty from the point of view of classical mechanics. From Newton's principles, the principle of relativity stated that two systems are described by the same physics if, relative to the first, the second is a uniformly moving coordinate system. In other words, invariant physical laws under Galilean transformations. Consequently, light is not aligned with the principle of relativity, as both observers will measure the same speed for it. There is no concept of relative speed. Thus, it is necessary to drop the idea of constant speed of light or the principle of relativity if one expects unity. However, Lorentz showed that the constancy of the speed of light is a necessary consequence. This left only the latter to work on. Einstein's reformulation of the principle of relativity will later be known as principle of Special Relativity [3].

Einstein started explaining the concept of simultaneity [2]. An a priori notion of simultaneity depends on the observer. Einstein example is the following: consider two lightnings that land at a certain distance from each other. An observer standing right in the middle point of the separation between the lightnings will perceive them simultaneously. However, if the same happens from someone moving at a certain velocity at the moment they land, he will be running toward one of them, and therefore will observe that one first. This consideration completely undermines space and time as absolutes. Imagine, for example, someone pursuing a long train. Since the observer is moving toward the train, it means that after receiving the light rays from the closest section of it, he will travel toward the furthermost part for a certain period effectively observing a train shorter in length compared to someone observing at rest. In other words, the concepts of time and space are flexible but not only that, time and space are intrinsically connected. In this way, Einstein restored the principle of relativity because he knew that the physical laws must be the same for all observers moving at constant velocity, as Newton and Galileo did in their moment. With the Pythagorean theorem, he displayed that physics are invariant under Lorentz transformations when the speeds involved are a non-negligible fraction of the speed of light. Surprisingly, the failure of Michelson-Morley experiment, which was dedicated to detect ether fluctuation and inspired physicist of the caliber of Lorentz and Poincaré, did not seem to have influenced Einstein thoughts. Poincaré was already in the path to derive its own relativity theory when Einstein pioneered Special Relativity and shared his ideas.

Later, Einstein sent a sequel with another fundamental reformulation of physics laws. At that time, scientists believed that the law of conservation of energy applied to the mass. However, Einstein showed that energy and mass are related according to his, probably most famous, formula: $E = mc^2$ [4], where E is the energy, m the mass and c the speed of light. Later on, Minkowski grasped the importance of Lorentz transformations and made the link to geometry, placing the time as an extra coordinate and constructing what we call *spacetimes* [5].

In search of a theory of gravity, Einstein realized that someone in free fall inside an isolated elevator has no way to know if it is because of gravity or another acceleration source, like mounting a rocket. In other words, he needed to construct a theory where all accelerated systems obeyed the same physics. Einstein realized that Euclidean geometry was not enough to write a gravity theory consistent with this observations. Up to here, it was understood that gravity acts in all massive objects. Now that energy and mass are related, it is natural to ask how light behaves in the presence of a gravitational field. Einstein proposed that light is composed of energy packets and explained the photoelectric effect [6]. Also, he predicted that the apparent position of stars will move when they are near the sun [7].

In regards to non-Euclidean geometry, in the 19th century, Gauss was one of the firsts to develop the field and he defined a curvature for two-dimensional surfaces embedded in tree dimensions. His apprentice, Riemann, generalized the concept of distance between two points in curved spaces and extended Gauss notions to higher dimensions. Riemann discovered a more general form of the Pythagoras' theorem, the concept of metric. Riemann, motivated by physics, insisted that real materials are mostly non-Euclidean at small distances. He noticed that the concept of force may be a manifestation of geometry. Riemann ideas were close to a geometric theory of gravity, however, he only worked with space and missed dealing with time. Einstein, while looking for tools for an unification of gravity and special relativity, remembered Gauss and Riemann's geometry and contacted his friend Grossmann to write his first version of General Relativity [7]. Einstein needed new mathematics and was introduced to tensor calculus. During the developments of his gravity theory, he exchanged letters with Hilbert and eventually achieved his breakthrough, under Hilbert hints, however, his field equations were still incorrect [8, 9]. He published almost at the same time with Hilbert for a correct General Relativity [10]. However, Einstein's approach was more pragmatic compared to the axiomatic approach of Hilbert [11]. Einstein's proposal of action involved a Lagrangian proportional to the square of the Christoffel symbol which does not transform as a tensor, while Hilbert suggested the Ricci scalar. Einstein used his field equations to compute the Mercury's perihelion precession [12], something no theory did before. However, his first computation was incorrect and he, later, reworked out the numerics. Also, he re-computed the deviation of the apparent position of stars with his corrected equations, a statement proven true by Eddington years later during an Eclipse [13].

After a consistent and covariant theory of gravity was developed, Einstein started to explore the cosmological implications of his claims. He predicted the existence of gravitational waves [14], the most important check nowadays of the validity of General Relativity. Later, he explored in detail the evolution of the universe and arrived to the conclusion that the universe could not be static [15]. However, Einstein was convinced of the universe's staticity and introduced a term to fix the issue: the cosmological constant, a fact he will call his *biggest blunder*. Einstein introduced the cosmological constant as source of negative pressure in the whole spacetime with the intention of balancing the gravitational effects. Nonetheless, Hubble, some years later, found evidence of an expanding universe and Friedmann realized that Einstein's static universe was unstable [16]. Friedmann's work evolved an universe with constant mass density and layed the first stone of modern cosmology.

Weeks after Einstein published his final consolidation of his gravity theory [17], Schwarzschild computed a exact solution of his field equations under the assumption of spherical symmetry [18]. Schwarzschild was interested in the simple problem of a massive particle and how space curved in response. It was the debut of *event horizons*, zones were the escape velocity was higher than c. However, physicist wanted a theory beyond Einstein's and looked for generalizations. Vermeil [19] showed that the scalar curvature was the only invariant compatible with General Relativity, even when more dimensions are added. Reissner, Weyl and Nordström, independently, worked out the case of Einstein gravity with Maxwell fields [20]. Cartan relaxed the condition of the spacetime to be Riemannian and introduced torsion [21]. Among the numerous developments in the field, Weyl in his inexhaustible search of unifying gravity and electrodynamics, included terms quadratic in the curvature [22]. Later, Bach studied in more detail this contribution and computed the corrections obtained in the dynamics [23]. Next, Lanczos proposed to include second powers of the curvature with different structures using Levi-Civita symbols and showed that a precise combination of certain terms does not contribute to the dynamics: the Gauss-Bonnet term [24]. The three –Weyl, Bach and Lanczos– worked in four dimensions and it was not until almost half a century later that Lovelock proposed a generalization for higher dimensions [25]. He developed a set whose equations of motion are symmetric, conserved and second order in the metric, in the same fashion as Einstein field equations. In four dimensions, Einstein gravity is the final implementation of Lovelock's scheme. However, as one raises the dimension, more structures can be added, which are quadratic in the curvature in five dimensions, cubic in the curvature in seven dimensions, and so on. Subsequently, he presented the Lagrangians whose variations lead to those equations, which are quasi-linear in second derivatives of the metric. From this property, some issues appear when defining the canonical momentum, a fact we will visit in chapter 2.

Even though Lovelock gravity is a natural extension of Einstein gravity for higher dimensions, based on the nature of the field equation, in principle, there was no reason to favor it over General Relativity. However, an increasing influence of string theory and the fact that the low energy effective actions of string theory predicted higher curvature terms dragged attention [26–28]. Also, quantum physics studies eventually lead to attempts to unify gravity with quantum field theory, that failed in General Relativity [29]. Higher powers of the curvature were among the candidates for success in renormalization [30]. Nowadays, Lovelock gravity play a role in adding corrections for holographic computations in the context of the anti-de Sitter/Conformal Field Theory duality [31].

After Newton's *Principia*, authors of the caliber of Euler, Lagrange, Jacobi and Hamilton improved and axiomatized Classical Mechanics [32, 33]. Lagrange, using the Euler's calculus of variations, developed a scheme to solve the dynamics of conservatives systems: the principle of least action. He broke the obsession of physicist of trying to describe everything with a unique set of equations and proposed a unique scheme to solve different problems. For example, the least action principle was a tool used by Hilbert when he wrote his first Lagrangian for General Relativity [11]. This computation and many others in the coming years were mostly based in Lagrangian Mechanics. However, the emergence of Quantum Mechanics brought the focus to Hamiltonian formalism. This formulation allowed to put momentum and positions at the same footing and compute phase spaces [33]. Also, Poisson brackets can often jump to commutator in Quantum Mechanics and are ideal for quantization [34]. These facts, and more, motivated the physicist around the middle of the 20th century to treat gravity with the magnifying glass of Hamiltonian Mechanics.

Arnowitt, Deser and Misner developed a scheme to decompose the four-dimensional Einsteinian gravity in three dimensional slices evolving in time, a frame later called the three plus one formalism [35, 36]. In this regime, the sample manifold is foliated in Cauchy surfaces (constant time) whose evolution in time is given by two new functions: the *Lapse* and *Shift*. Therefore, the original variable, the metric, is split between the co-dimension one metric and the Lapse and Shift functions. Also, it can be shown that the Lapse and Shift functions are auxiliary fields without dynamics, which leave only the co-dimension one metric with conjugate momentum [35, 36]. The authors also computed a formula for the energy of the system [35]. The drawback is that the covariance in the full spacetime is no longer evident, leaving explicit covariance for the co-dimension one hypersurfaces. Later, the Hamitonian form for Lovelock Lagrangians was given by Teitelboim and Zanelli [37]. Both works assume the time coordinate as evolution parameter. However, Einstein gravity and its modifications quoted here reside in a spacetime, whose purpose is to treat time and space in the same footing. Thus, the aim of this work is to show that a study of spatial evolution naturally leads to the Hamiltonian of the system and junction conditions for discontinuous distributions of matter.

Conserved quantities in physics are important to understand the dynamics of a system. Something conserved will remain the same, no matter what happens and are usually connected to observables, making them essential in experimental physics. We can write down the equation quantities and use those equations to understand how other variables will change with different conditions. In gravity, a considerable portion of research have been spinning around the computation of conserved quantities or finding formulas to obtain them. At the beginning of the 20th century, after the release of Einstein's Special Relativity, Herglotz exhibited the relationship between the Poincaré group and the conservation laws of Special Relativity by using variational techniques [39]. Later, Engel, motivated by his tutor Klein, analyzed Galilean symmetries and connected it to Lie's theory [40] and the classical conservation laws [41]. Based on both authors, Emmy Noether released an elegant paper linking symmetries, variational principle and conserved quantities [42]. She showed that a differentiable *global* symmetry of an action for a physical system –with conservative forces– has a corresponding conservation law, which is today known as Noether's first theorem, a workhorse of nowadays physics. Also, she showed that a *local* symmetry implies the dynamics admits a nontrivial differential relation, a statement known as Noether's second theorem, normally used in gauge theory (See e.g. Ref.[43]). Her work, however, requires a well defined variational principle. Fortunately, boundary terms do not modify the equations of motion and therefore can be freely added to adapt to the preferred boundary conditions. In particular, Dirichlet boundary conditions, that prescribe a fixation of the variable values –but not their derivatives– at the boundary is specially attractive for the simplicity to define the energy-momentum tensor and the connection with Hamiltonian mechanics. The Einstein-Hilbert Lagrangian, which is just the Ricci scalar, produces, for arbitrary variations, terms that are proportional to variations of the metric and its first derivatives, a fact that brought intrigue for boundary terms. Gibbons and Hawking –and independently York– showed the boundary term that allows to study the Dirichlet problem for gravity [44, 45]. Later on, Myers achieved the same goal for Lovelock gravity [46] (See also Refs.[37, 47]). In this work, we will show that the boundary terms proposed by Myers are linked to writing Lovelock gravity as a functional purely of the metric and his first derivatives. We will elaborate further in this point in Chapter 2.

If gravity and geometry are intertwined, one of the problems one may explore is what happens near an interface. In elemental electrostatics, it is understood that the existence of a charged plate gives rise to a jump in the electric field [48]. The authors Lanczos and Darmois, around the end of the first quarter of the 20th century, are mentioned as pioneers in this field for gravity [49] (See also Ref.[50] and references therein). A mathematical understanding of the formation of celestial bodies was the driving force behind those and several subsequent developments. Oppenheimer and Snyder's model provided en example that served as guide to the collapse problem in General Relativity [51]. It simply assumes a collapsing homogeneous dust cloud of finite mass surrounded by a vacuum exterior. Even nowadays, studies in the Oppenheimer-Snyder scenarios see the light of day frequently [52]. However, it was past the middle of the 20th century when authors like Misner and Sharp, in Gravitational Collapse [53], and, in particular, Israel, in his work in hypersurfaces [54], gave the fundamentals to the nowadays understanding of junction conditions in General Relativity. At this point, authors started to talked about the second fundamental form or extrinsic curvature, both related to first derivatives of the metric. In contrast with the Oppenheimer-Snyder scenario, Israel considered an infinitesimally thin shell, which later was studied under the tag *Thin Shell Collapse*. As in General Relativity, Davis derived the junction conditions but now for Einstein-Gauss-Bonnet gravity, the simplest Lovelock Lagrangian beyond Einstein's[2] [56] (See also Refs.[57, 58, 61]). Nonetheless, junction conditions were also involved in brane-world, a cosmological model whose purpose was to solve the hierarchy problem[3] [60]. Corrections for these models with higher orders in the curvature were included later on [57, 61]. Studies on junction conditions not only included Lovelock terms, a wide fauna of possible tensorial combinations were explored, for example, f(R) theories [49]. In particular, the treatment of moving thin shells in General Relativity, as depicted in Refs.[62, 63] is specially relevant for this work. Their conclusions were extended to Lovelock theories in Ref.[64], however, in Chapter 3, we will show their treatment is mislead and not consistent with the variational principle.

[2]In the case of asymptotically locally anti-de Sitter spacetimes in four dimensions –associated to gravity with negative cosmological constant– it has been proven that the addition of the Gauss-Bonnet density, which is locally a boundary term, is enough to obtain finite conserved charges for a precise coupling constant [55].

[3]The difference between the orders of magnitude of Plank and electroweak energy scales is an obstruction for attempts of unification (See e.g. Ref.[59]).

Chapter 2

First-order Lagrangian and Hamiltonian of Lovelock gravity

2.1 Introduction

In the first chapter, we analyzed the example of a Lagrangian linear in the acceleration and clarified that the addition of suitable boundary terms can reduce it to a first-order action functional compatible with the Dirichlet problem. In order to avoid misconceptions, we must first clarify what is the meaning of *first-order* in the context of gravity. Since there are several variables and the building blocks are the Riemann tensor, we expect a term linear in the acceleration for each coordinate. Therefore, first-order makes reference to a functional whose derivatives with respect to the *evolution* coordinate are, at most, of order one. In accordance with the Gaussian frame A.10, this variable is labelled as r. Even though most of the computations here assume spatial foliation, it is always possible to switch to temporal foliation using the change of variables at the end of the above chapter. The assumptions made in the Classical Mechanics example of Chapter 1 were aimed to mimic the properties of the algebraic structures of pure gravity. The equations of motion of Einstein gravity, or more generally, in higher dimensions, Lovelock gravity [25], are of second order and it hints that a description in terms of a first-order

Lagrangian should be attainable. Also, the fact that the variation of the Dirichlet action leads to the canonical momentum times variations of the boundary metric at the boundary was shown in Ref.[69], reinforcing the hypothesis.

Certainly, this is true for General Relativity, whose Lagrangian is the Ricci scalar, as shown by Arnowitt, Deser, and Misner (ADM) [35, 36] in their work written in 3+1 formalism. The final expression is wirtten in terms of the extrinsic and intrinsic curvatures for a time foliation. The Lovelock terms are also linear in second derivatives by construction, however, second derivatives with respect to a certain variable may be multiplied by second derivatives with respect to others, a property refereed as quasi-linearity. This fact can be checked explicitly when Lovelock terms are written as a contraction of curvatures and antisymmetric symbols[1], where the former gives a derivative for each component and the latter allows to have at most each component twice. The Lovelock first-order Lagrangians (and corresponding Hamiltonian) were found by Teitelboim and Zanelli [37, 38].

In order to prospect a simpler path to reproduce said results, the structure of the boundary contributions must be thoroughly explored. In General Relativity, the boundary term that recovers a Lagrangian compatible with Dirichlet boundary conditions is the Gibbons-Hawking-York (GHY) term [44, 45]. Later, its generalization to Lovelock theories was obtained by Myers [46], see also [47]. In this section, the procedure to recover first-order Lagrangians is depicted, and at the same time, a decomposition of each Lovelock term between a bulk and a boundary term is shown. The main tool for the process is employing the Stokes theorem to write the Myers terms as bulk terms, a process here named *bulkanization*.

[1]See Eq.(2.35) for details

2.2 The first-order Lagrangian and Hamiltonian in Classical Mechanics

In Classical Mechanics, a first-order Lagrangian by definition is a Lagrangian function that depends on at most first-order derivatives of the generalized coordinates with respect to a certain evolution variable. Throughout this section, this variable will be the time coordinate t, as temporal evolution is the most common case.

The principle of least action dictates that the dynamics of a physical system can be determined by computing the stationary value of the line integral of its Lagrangian (called action integral), where the Lagrangian $L[q,\dot{q}]$ includes all energies involved in the system [32]. Mathematically, in a sample time interval $t=[t_i,t_f]$, it requires to compute the minimum of the action integral $I[q,\dot{q}]$, such that

$$\delta I[q,\dot{q}] = \int_{t_i}^{t_f} dt\, \delta L[q,\dot{q}] = 0\,. \tag{2.1}$$

Here, q labels the generalized coordinates and a dot stands for derivatives with respect to t. We evoke the chain rule in the form

$$\delta L[q,\dot{q}] = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\,, \tag{2.2}$$

to work out Eq.(2.1) and get

$$\delta I[q,\dot{q}] = \int_{t_i}^{t_f} dt\, \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q + \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_i}^{t_f}\,. \tag{2.3}$$

The result shows two expressions that must be zero independently, one in the bulk and the other at the boundary. Because we want to find the extremum path for the action from within all possible configurations, δq must be different from zero in the bulk and therefore we recover the, so called, Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0\,. \tag{2.4}$$

On the other hand, we need boundary conditions to cancel the second term in Eq.(2.3). There are several options for boundary conditions, among the we have:

Dirichlet, that implies to supply values for q at the boundary, Neumann, which requires values for $\partial L/\partial\dot{q}$ at the boundary, Robin, that fixes a linear combination of q and $\partial L/\partial\dot{q}$, Mixed, that implies different boundary conditions for sub-segments of the boundary, etcetera. We will focus our attention in Dirichlet boundary conditions, characterized by $\delta q(t_i) = \delta q(t_f) = 0$. The dynamics given by Eq.(2.4) is of second-order in derivatives with respect to t, as long as $\partial L/\partial\dot{q}$ is a function of the velocity. This is always the case, for example, for Lagrangians that include a kinetic term proportional to the square of the velocity. Lagrangians linear in the velocity, corresponding to the so-called canonical actions, are out the scope of this work and are not discussed here. However, we are interested in what happens to the dynamics if we add a term linear in the acceleration.

Consider now a first-order Lagrangian supplemented with a term linear in second derivatives

$$\bar{L}[q.\dot{q},\ddot{q}] = L[q,\dot{q}] + \ddot{q}F(q,\dot{q})\,, \tag{2.5}$$

where $F(q,\dot{q})$ is any function of positions and velocities satisfying $\partial F/\partial\dot{q} \neq 0$ (otherwise one can eliminate $\ddot{q}$ by integration by parts). The variation of the new piece $\ddot{q}F(q,\dot{q})$ is

$$\delta\big(\ddot{q}F(q,\dot{q})\big) = \frac{d}{dt}\left[\delta\dot{q}F\right] - \delta\dot{q}\frac{dF}{dt} + \ddot{q}\left[\frac{\partial F}{\partial q}\delta q + \frac{\partial F}{\partial\dot{q}}\delta\dot{q}\right]\,, \tag{2.6}$$

The variation of the line integral for the full Lagrangian (2.5) is then given by

$$\delta\bar{I} = \int_{t_i}^{t_f} dt\,\left\{\left[\ddot{q}\frac{\partial F}{\partial q} + \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}}\right)\right]\delta q + \left[\ddot{q}\frac{\partial F}{\partial\dot{q}} - \frac{dF}{dt}\right]\delta\dot{q}\right\} + \left[\frac{\partial L}{\partial\dot{q}}\delta q + F\delta\dot{q}\right]_{t_i}^{t_f}\,.$$

Now the problem requires also initial conditions for $\dot{q}$. Unfortunately, this requirement is in general incompatible with the fact that the equations of motion are still of second order, giving two integration constants. Initial conditions for position and velocity of the initial and final configurations require four inputs if we pursue Dirichlet boundary conditions. This problem happens because $\bar{L}$ depends on the acceleration $\ddot{q}$. In order to solve this issue, we will subtract a boundary term, namely B, that depends on position and velocity

$$L_1 = \bar{L} - \frac{d}{dt}B[q,\dot{q}]\,. \tag{2.7}$$

Boundary terms do not modify the dynamics as they do not contribute to variations living in the bulk in the line integral. However, they modify the boundary terms, changing the boundary conditions needed. In order to search for the form of B, we can replace the chain rule for the temporal derivative of B

$$\frac{dB}{dt} = \frac{\partial B}{\partial q}\dot{q} + \frac{\partial B}{\partial \dot{q}}\ddot{q}\,, \tag{2.8}$$

in the last expression to obtain the Lagrangian

$$\bar{L}_1 = L[q,\dot{q}] - \frac{\partial B}{\partial q}\dot{q} + \ddot{q}\left(F(q,\dot{q}) - \frac{\partial B}{\partial \dot{q}}\right)\,. \tag{2.9}$$

The factor next to $\ddot{q}$ gives a differential equation for B that after solving allows to remove the acceleration dependence in L_1. In other words, if we pick B such that

$$F(q,\dot{q}) - \frac{\partial B}{\partial \dot{q}} = 0 \Rightarrow B = \int F(q,\dot{q})d\dot{q}\,, \tag{2.10}$$

we comeback to a first-order Lagrangian. The same conclusion can be obtained from the point of view of the, on-shell, variational principle for the line integral of L_1

$$\delta I_1 = \left[\left(\frac{\partial L}{\partial \dot{q}} - \frac{\partial B}{\partial q}\right)\delta q + \left(F - \frac{\partial B}{\partial \dot{q}}\right)\delta\dot{q}\right]_{t_i}^{t_f}\,. \tag{2.11}$$

The same factor is found next to variations of $\dot{q}$. Then, solving Eq.(2.10) allows a well-posed Dirichlet problem, or equivalently, renders the Lagrangian to first-order in derivatives with respect to the evolution variable.

For a Lagrangian of this nature, the Hamiltonian is defined by the Legendre transform

$$H[p,q] = p\dot{q}(p) - L[q,\dot{q}(p)] \tag{2.12}$$

where

$$p = \frac{\partial L}{\partial \dot{q}}\,, \tag{2.13}$$

is the canonical momentum conjugate to the canonical variable q with the time as evolution parameter. Notice that computing the Hamiltonian as a function solely of p and q requires $\dot{q}$ to be a function of p. As we mentioned before, this fact is secured by Lagrangians at least quadratic in the velocity and is mathematically encoded in the condition

$$\frac{\partial^2 L}{\partial \dot{q}^2} \neq 0\,. \tag{2.14}$$

From a physical point of view it is natural to require the Lagrangian to be an even power of the velocity, as it secures that the Hamiltonian, that is often also the energy of the system, to be bounded from below.

The dynamics can be derived matching both sides in the variation of Eq.(2.12)

$$\frac{\partial H}{\partial p}\delta p + \frac{\partial H}{\partial q}\delta q = \delta p \dot{q} + p \delta \dot{q} - \frac{\partial L}{\partial q}\delta q - \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \tag{2.15}$$

that combined with Eq.(2.4) yields

$$\frac{\partial H}{\partial p} = \dot{q} \qquad \frac{\partial H}{\partial q} = -\dot{p}\,. \tag{2.16}$$

In addition, in these variables the Euler-Lagrange equation (2.4) can be rewritten as

$$\frac{\partial L}{\partial q} - \dot{p} = 0\,. \tag{2.17}$$

The above equation makes manifest that the second order derivatives in the equations of motion are all encoded in derivatives of the momentum. We will exploit this fact to compute junction conditions in the next section.

2.3 The first-order Lagrangian and Hamiltonian of Einstein gravity

This brief review of Einstein gravity and its Hamiltonian formalism intends to design a logic line to study Lovelock gravity. This last, shares key properties with General Relativity such as attainable first-order form, that can be treated using Classical Field Theory tools [68].

In Lagrangian description, the starting point is the Einstein-Hilbert action functional

$$I_{\text{EH}}[g] = \frac{1}{16\pi G}\int\limits_{\mathcal{M}} d^D x \sqrt{-g}\big(R - 2\Lambda\big)\,, \tag{2.18}$$

where D is the dimension of the spacetime manifold $\mathcal{M}$, G is Newton's gravitational constant, R in the Ricci scalar[2], g is the determinant of the metric $g_{\mu\nu}$ and Λ is the cosmological constant. This last, determines the local curvature value of the background spacetime, that is, when all global physical properties, such as the mass, angular momentum or electric charge, are set to zero.

In the previous chapter, the evolution coordinate selected was the time coordinate t. However, in this section, radial evolution is explored. More explicitly, in the radial foliation frame (A.10), the 'velocity' of h_{ij} is $h'_{ij} \propto K_{ij}$ and the acceleration h''_{ij} is related to K'_{ij}. The Lagrangian (2.18) is linear in the acceleration, in similar fashion as the first-order Lagrangians considered in the previous chapter. Thus, the first term in (2.22), in the frame (A.10), can be rewritten as

$$\sqrt{-g}R = \sqrt{-g}\big(\mathcal{R} - K^2 - K^i_j K^j_i\big) + 2\sqrt{-h}K' \,. \tag{2.19}$$

where a prime stands for a radial derivative. It is clear that radial derivatives of the extrinsic curvature contain a term linear in second radial derivatives. With the relation $(\sqrt{-h})' = -K\sqrt{-g}$ the integrand in Eq.(2.19) can be cast in the form

$$\sqrt{-g}R = \sqrt{-g}\big(\mathcal{R} + K^2 - K^i_j K^j_i\big) + 2\big(\sqrt{-h}K\big)' \,. \tag{2.20}$$

Here, the bulk integral was split into a quantity that does not contain second normal derivatives of the metric plus a total derivative. Consequently, the second-order terms were packed in a total derivative that does not modify the dynamics and play a role in the boundary conditions.

Boundary conditions are required to appropriately formulate the variational principle. Consider, for example, the variation of the action (2.18) given by

$$\delta I_{\text{EH}} = \frac{1}{16\pi G}\int\limits_{\mathcal{M}} d^Dx\sqrt{-g}\left((G_{\mu\nu} + \Lambda g_{\mu\nu})\delta g^{\mu\nu} + \nabla_\mu\left(g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\mu\alpha}\delta\Gamma^\beta_{\alpha\beta}\right)\right), \tag{2.21}$$

where $G_{\mu\nu}$ is the Einstein tensor, $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols and ∇_μ is the covariant derivative[3]. The second term on the r.h.s. of (2.21) –which can be evaluated

[2] For conventions, see Appendix A

[3] For conventions, see Appendix A

using Gauss' theorem, on the $d = D-1$ dimensional boundary $\partial\mathcal{M}$ of $\mathcal{M}$– contains both zero-th and first order derivatives of the metric. However, the equations of motion described by the Einstein tensor are of second order and provide two constants of motion. This information is insufficient to fix simultaneously the zero-th and first order derivatives at the boundaries. In other words, we have to impose further restrictions or add boundary terms that renders the result suitable for the desired boundary conditions.

For example, the boundary term in Eq.(2.21) for the Gaussian coordinate system (A.10) with Christoffel symbols (B.1) includes a term δK, where $K = h^{ij}K_{ij}$, making manifest the variation of the normal derivative of h_{ij}. In order to connect to the Classical Mechanics example in the above chapter, we are interested in Dirichlet boundary conditions, that in this setup imposes $\delta g_{\mu\nu} = 0$ at the boundaries.

Hence, the action principle based on Dirichlet boundary conditions can be achieved if the action (2.18) is supplemented with the Gibbons-Hawking-York (GHY) boundary term [44, 45] to construct the Dirichlet action

$$\tilde{I}_{\mathrm{EH}}[g] = \frac{1}{16\pi G}\int\limits_{\mathcal{M}} d^D x\sqrt{-g}[R-2\Lambda] - \frac{1}{8\pi G}\int\limits_{\partial\mathcal{M}} d^d x\sqrt{-h}K\,, \tag{2.22}$$

where $D = d+1$. Notice that the GHY term cancels the boundary term that appears as a total derivative in Eq.(2.20). The new action is therefore a bulk functional with at most first-order derivatives with respect to r. Furthermore, this shows that if one forgets about boundary conditions and drop boundary terms from the Einstein-Hilbert action, one arrives to the same dynamics. However, this is a mere accident, as not discussing boundary conditions does not secure that the obtained solutions fulfill the principle of least action.

At the end, the Lagrangian of the Dirichlet action, written in first-order form $\mathcal{L}^{(1)}$, matches the well-known Arnowitt-Deser-Misner Lagrangian (ADM) [35, 36] and provides the action integral

$$\tilde{I}_{\mathrm{EH}}[g] \;=\; \frac{1}{16\pi G}\int\limits_{\mathcal{M}} d^D x\sqrt{-h}N\big(\mathcal{R}+K^2-K^i_jK^j_i-2\Lambda\big)$$

$$= \frac{1}{16\pi G}\int\limits_{\mathcal{M}} d^D x\,\mathcal{L}^{(1)}\,. \tag{2.23}$$

One can see that N is an auxiliary field (a field without evolution), while the field h_{ij} appears in the typical form of the velocity square (${h'_{ij}}^2 \propto {K_{ij}}^2$). The variation of the above action leads, on-shell, to

$$\delta\tilde{I}_{\mathrm{EH}} = \frac{1}{16\pi G}\int\limits_{\partial\mathcal{M}} d^d x\,\pi^{ij}\delta h_{ij}\,, \tag{2.24}$$

where the factor of the variation of the boundary metric is

$$\pi^{ij} = \sqrt{-h}(K^{ij} - Kh^{ij})\,. \tag{2.25}$$

It is clear that the action principle constructed from $\mathcal{L}^{(1)}$ is compatible with the Dirichlet boundary condition, as required.

It can be explicitly checked that variations with respect to N and h_{ij} of the above Lagrangian yield respectively to the constraint $G^r_r = 0$ and the dynamical component $G^i_j = 0$ of the equations of motion written in Gaussian coordinates.

In addition, the missing dynamical information of the full metric $g_{\mu\nu}$ due to the gauge-fixing (A.10) (without the $G^i_r = 0$ components) can be recovered introducing the full ADM decomposition

$$ds^2 = N(r,x^i)^2 dr^2 + h_{ij}(dx^i + N^i(r,x^i)dr)(dx^j + N^j(r,x^i)dr)\,, \tag{2.26}$$

where $N(r,x^i)$ is the lapse function and $N^i(r,x^i)$ is the shift function, both of them auxiliary fields, through the redefinition of the extrinsic curvature

$$K_{ij} = -\frac{1}{2N}(\partial_r h_{ij} - \bar{\nabla}_i N_j - \bar{\nabla}_j N_i)\,, \tag{2.27}$$

where $\bar{\nabla}_i$ is the covariant derivative associated to h_{ij}.

With these results at hand, it is natural to ask how can they be seen in the Hamiltonian formalism. In fact, the tensor π_{ij}, in resemblance with Classical Mechanics, satisfy

$$\pi^{ij} = \frac{\partial\mathcal{L}^{(1)}}{\partial h'_{ij}}\,, \tag{2.28}$$

which in Hamiltonian formalism corresponds to the canonical momentum associated to h_{ij}. Then, the next goal is to succinctly review the Hamiltonian approach of General Relativity to pave the intuition for the Lovelock gravity case.

In order to obtain the Hamiltonian density, it is necessary to express the velocity –or extrinsic curvature– as a function of the canonical momentum. Hence, inversion of the relation (2.25) leads to

$$K_{ij}=\frac{1}{\sqrt{-h}}\left(\pi_{ij}-\frac{1}{d-1}\pi h_{ij}\right)\,, \tag{2.29}$$

where $\pi=\pi^{ij}h_{ij}$ is the trace of the momentum taken with the induced metric. Therefore, the Hamiltonian density $\mathcal{H}=\pi^{ij}h'_{ij}-\mathcal{L}^{(1)}$ in terms of the extrinsic curvature and canonical momentum, respectively, has the form

$$\begin{aligned}\mathcal{H} &= -N\sqrt{-h}\left\{\mathcal{R}-K^2+K^i_jK^j_i-2\Lambda\right\} \\ &= -N\sqrt{-h}\left\{\mathcal{R}+\frac{1}{-h}\left(\pi^i_j\pi^j_i-\frac{1}{d-1}\pi^2\right)-2\Lambda\right\}\,. \end{aligned} \tag{2.30}$$

However, there is still information missing due to the Gauss-normal frame. In order to solve this issue, Eq.(2.27) is used to find a new expression for h'_{ij} for the full ADM decomposition. The final Hamiltonian density is then

$$\mathcal{H}=N\mathscr{H}+N^i\mathscr{H}_i\,, \tag{2.31}$$

where N and N^i are Hamiltonian multipliers and the corresponding Hamiltonian constraints $\mathscr{H}$ and $\mathscr{H}_i$ take the form

$$\begin{aligned}\mathscr{H} &= -\sqrt{-h}\left(\mathcal{R}+\frac{1}{-h}\left(\pi^i_j\pi^j_i-\frac{1}{d-1}\pi^2\right)-2\Lambda\right)\,, \\ \mathscr{H}_i &= -2\bar{\nabla}_j\pi^j_i\,. \end{aligned} \tag{2.32}$$

For completeness, the dynamical equivalence between both (Lagrangian and Hamiltonian) descriptions is explored. From the action (2.22), the equations of motion are linked to the Einstein tensor. In the Gauss-normal coordinate frame, written in terms of the extrinsic curvature and canonical momentum respectively, we have three types of equations, namely,

$$G^r_r = \mathcal{R}-K^2+K^i_jK^j_i$$

$$
\begin{aligned}
&= \mathcal{R} + \frac{1}{-h}\left(\pi^i_j \pi^j_i - \frac{1}{d-1}\pi^2\right) , \\
G^r_i &= \frac{1}{N}\left(\bar{\nabla}_i K - \bar{\nabla}_k K^k_i\right) \\
&= -\frac{1}{N\sqrt{-h}} \bar{\nabla}_k \pi^k_j , \\
G^{ij} &= \mathcal{G}^{ij} - K\big(K^{ij} - K h^{ij}\big) + \frac{1}{2}\big(K^k_l K^l_k - K^2\big) h^{ij} + \frac{1}{N}(K^i_k - K\delta^i_k)' h^{kj} \\
&= \mathcal{G}^{ij} + \frac{1}{2(-h)}\left(\pi^{kl}\pi_{kl} - \frac{1}{d-1}\pi^2\right) h^{ij} + \frac{1}{N\sqrt{-h}}\big(\pi^{ij}\big)' \\
&\qquad - \frac{2}{-h}\left(\pi^i_k \pi^{kj} - \frac{1}{d-1}\pi\pi^{ij}\right) ,
\end{aligned}
$$

where $\mathcal{G}^{ij}$ is the Einstein tensor computed from the codimension one metric h_{ij}. With this information, it is straightforward to show that the Hamiltonian [68] and Lagrangian dynamics are equivalent according to the correspondence

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial N} = 0 \quad &\Leftrightarrow \quad G^r_r + \Lambda = 0 \, , \\
\frac{\partial \mathcal{H}}{\partial N^i} = 0 \quad &\Leftrightarrow \quad G^r_i = 0 \, , \\
\frac{\partial \mathcal{H}}{\partial \pi^{ij}} = \big(h_{ij}\big)' \quad &\Leftrightarrow \quad K_{ij} = -\frac{1}{2N}\big(h_{ij}\big)' , \\
\frac{\partial \mathcal{H}}{\partial h_{ij}} - \partial_k \frac{\partial \mathcal{H}}{\partial(\partial_k h_{ij})} + \partial_l \partial_k \frac{\partial \mathcal{H}}{\partial(\partial_l \partial_k h_{ij})} \quad &= \quad -\big(\pi^{ij}\big)' \Leftrightarrow G^{ij} + \Lambda h^{ij} = 0 \, .
\end{aligned} \tag{2.33}
$$

Notice that the relations $G^r_r + \Lambda = \mathscr{H}/2\sqrt{|h|}$ and $G^r_i = \mathscr{H}_i/2N\sqrt{|h|}$ are results that must be satisfied for consistency. In pure gravity, the variable is the spacetime itself and the information not contained in the d-dimensional slices of the ADM decomposition is input through geometric constrains.

2.4 First-order Lagrangian and Hamiltonian formalism for Lovelock theory

General Relativity is described by a Lagrangian linear in the curvature that, by definition, is linear in second derivatives of the metric and generate second-order field equations. When terms with higher powers of the curvature are added, it is possible to construct more Lagrangians that meet this criteria. In particular, Lovelock gravity [25], which is a precise combination of higher curvature terms, also has boundary terms that make the final Lagrangian apt for Dirichlet boundary condition [46]. These facts allow us to extend the treatment depicted above to this gravity theory and analyze the consequences.

The Lovelock gravity action is given by

$$I[g] = \frac{1}{16\pi G} \sum_{p=0}^{\lfloor \frac{D-1}{2} \rfloor} \alpha_p \int\limits_{\mathcal{M}} d^D x \mathcal{L}_p \,, \tag{2.34}$$

where $\{\alpha_p\}$ are the coupling constants, $\lfloor\cdot\rfloor$ is the floor function and each Lovelock term of degree p in the curvature is given by

$$\mathcal{L}_p = \frac{1}{2^p}\sqrt{-g}\delta^{[\mu_1\cdots\mu_{2p}]}_{[\nu_1\cdots\nu_{2p}]} R^{\nu_1\nu_2}_{\mu_1\mu_2}\cdots R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}} \,. \tag{2.35}$$

The reason the sum ends at $\lfloor (D-1)/2 \rfloor$ is because we account for all dynamical terms. From (2.35), one can see that if $2p > D$, then the Kronecker delta vanish since it becomes unavoidable to repeat a coordinate. On the other hand, the case $2p = D$ is special as $\mathcal{L}_p$ becomes a total derivative and therefore do not modify the dynamics. In fact, this can be explained with the formula for the Euler characteristic

$$\int_{\mathcal{M}} d^{2p}x\mathcal{L}_p = (4\pi)^p p! \chi(\mathcal{M}) + \int_{\partial\mathcal{M}} d^{2p-1}x\beta_p \tag{2.36}$$

where $\chi(\mathcal{M})$ is the Euler characteristic of the manifold $\mathcal{M}$ and the boundary contribution obeys

$$\beta_p = 2p\sqrt{-h}\int_0^1 ds\delta^{[i_1\cdots i_{2p-1}]}_{[j_1\cdots j_{2p-1}]} K^{i_1}_{j_1}\left(\frac{1}{2}\mathcal{R}^{j_2j_3}_{i_2i_3} - s^2 K^{j_2}_{i_2}K^{j_3}_{i_3}\right)\times$$
$$\times\left(\frac{1}{2}\mathcal{R}^{j_{2p-2}j_{2p-1}}_{i_{2p-2}i_{2p-1}} - s^2 K^{j_{2p-2}}_{i_{2p-2}}K^{j_{2p-1}}_{i_{2p-1}}\right)\,. \tag{2.37}$$

In this context, $\mathcal{L}_p$ is the p-th Euler density with β_p its corresponding correction due to the presence of a boundary.

Variations of the Lovelock Lagrangian lead –on-shell– to the contribution

$$\delta I = \frac{1}{16\pi G}\sum_{p=0}^{\lfloor \frac{D-1}{2} \rfloor} \frac{p\alpha_p}{2^{p-1}}\int_{\mathcal{M}} d^D x\sqrt{-g}\nabla_{\mu_1}\left(\delta^{[\mu_1\cdots\mu_{2p}]}_{[\nu_1\cdots\nu_{2p}]} g^{\nu_2\sigma}\delta\Gamma^{\nu_1}_{\sigma\mu_2} R^{\nu_3\nu_4}_{\mu_3\mu_4}\cdots R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right)$$

that contains variations of the Christoffel symbols that tamper Dirichlet boundary conditions.

Therefore, in the same line of reasoning as in GR, consider the Dirichlet action for a Lovelock gravity theory

$$\tilde{I}[g] = \frac{1}{16\pi G} \sum_{p=0}^{\lfloor \frac{D-1}{2} \rfloor} \alpha_p \left(\int\limits_{\mathcal{M}} d^D x \mathcal{L}_p - \int\limits_{\partial\mathcal{M}} d^d x \beta_p \right) , \tag{2.38}$$

where the corresponding $p-$th boundary term β_p –often called Myers term– renders the action suitable for Dirichlet boundary conditions [46, 47]. Notice that the action is the dimensional continuation of the Euler characteristic, in other words, the elements in Eq.(2.36) are employed in a dimension high enough to contribute to the dynamics ($2p < D$). Variations of the above action lead to the familiar form

$$\delta\tilde{I} = \frac{1}{16\pi G} \left(-\int\limits_{\mathcal{M}} d^D x \sqrt{-g} \mathcal{E}^{\mu\nu} \delta g_{\mu\nu} + \int\limits_{\partial\mathcal{M}} d^d x \, \pi^{ij} \delta h_{ij} \right) , \tag{2.39}$$

where the equations of motion for this Lagrangian are

$$\mathcal{E}^{\mu}_{\nu} = - \sum_{p=0}^{\lfloor \frac{D-1}{2} \rfloor} \frac{\alpha_p}{2^{p+1}} \delta^{[\mu\mu_1 \cdots \mu_{2p}]}_{[\nu\nu_1 \cdots \nu_{2p}]} R^{\nu_1\nu_2}_{\mu_1\mu_2} \cdots R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}} \, . \tag{2.40}$$

Furthermore, the canonical momentum π^{ij} has now the formula [69]

$$\pi^{ij} = \sum_{p=0}^{\lfloor \frac{D-1}{2} \rfloor} \alpha_p \pi^{ij}_{(p)} \, , \tag{2.41}$$

where the contribution coming from the p-th Lovelock density takes the form

$$\pi^{ij}_{(p)} = -p\sqrt{-h} \int_0^1 ds \, \delta^{[ii_1 \cdots i_{2p-1}]}_{[kj_1 \cdots j_{2p-1}]} h^{kj} K^{j_1}_{i_1} \left(\frac{1}{2} \bar{R}^{j_2 j_3}_{i_2 i_3} - s^2 K^{j_2}_{i_2} K^{j_3}_{i_3} \right) \times \cdots$$
$$\cdots \times \left(\frac{1}{2} \bar{R}^{j_{2p-2} j_{2p-1}}_{i_{2p-2} i_{2p-1}} - s^2 K^{j_{2p-2}}_{i_{2p-2}} K^{j_{2p-1}}_{i_{2p-1}} \right) . \tag{2.42}$$

In order to relate the Dirichlet Lagrangian to its first-order Lagrangian form and Hamiltonian, we need to perform two intermediate computations: i) expressing the bulk Lagrangian density $\mathcal{L}_p$ in the coordinate frame (A.10), ii) writing down the boundary term β_p as a bulk term, taking radial derivatives of the boundary quantities. This task is explicitly carried out in Appendix C. In particular, Eq.(C.14) implies

$$\int\limits_{\mathcal{M}} d^D x \mathcal{L}_p - \int\limits_{\partial\mathcal{M}} d^d x \beta_p = \int\limits_{\mathcal{M}} d^D x N \mathcal{H}_p + 2p \int\limits_{\mathcal{M}} d^D x \sqrt{-g} \int_0^1 ds \delta^{[i_1 \cdots i_{2p}]}_{[j_1 \cdots j_{2p}]} K^{j_1}_{i_1} K^{j_2}_{i_2} \times$$
$$\times \left(\frac{1}{2} \mathcal{R}^{j_3 j_4}_{i_3 i_4} - s^2 K^{j_3}_{i_3} K^{j_4}_{i_4} \right) \times \cdots \times \left(\frac{1}{2} \mathcal{R}^{j_{2p-1} j_{2p}}_{i_{2p-1} i_{2p}} - s^2 K^{j_{2p-1}}_{i_{2p-1}} K^{j_{2p}}_{i_{2p}} \right), \quad (2.43)$$

where the tensor $\mathcal{H}_p$ satisfies the equation

$$\mathcal{H}_p = \frac{1}{2^p} \sqrt{-h} \delta^{[i_1 \cdots i_{2p}]}_{[j_1 \cdots j_{2p}]} R^{j_1 j_2}_{i_1 i_2} \times \cdots \times R^{j_{2p-1} j_{2p}}_{i_{2p-1} i_{2p}} . \quad (2.44)$$

In Eq.(2.43), the tensor quantities in the r.h.s have been conveniently packed to highlight the connection with the Hamiltonian formalism, as will be clarified later on. The Dirichlet Lagrangian after employing Eq.(2.43) to get rid of the term linear in acceleration and boundary terms leads to the first-order Lagrangian

$$\mathcal{L}^{(1)} = \sum_{p=0}^{\left\lfloor \frac{D-1}{2} \right\rfloor} \alpha_p \mathcal{L}^{(1)}_p \quad (2.45)$$

where $\mathcal{L}^{(1)}_p$ is a polynomial of the boundary Riemann tensor $\mathcal{R}^{ij}_{kl}$, the extrinsic curvature K_{ij} and the boundary metric h_{ij}, such that

$$\mathcal{L}^{(1)}_p = N \mathcal{H}_p + 2p \sqrt{-g} \int_0^1 ds \delta^{[i_1 \cdots i_{2p}]}_{[j_1 \cdots j_{2p}]} K^{j_1}_{i_1} K^{j_2}_{i_2} \times$$
$$\times \left(\frac{1}{2} \mathcal{R}^{j_3 j_4}_{i_3 i_4} - s^2 K^{j_3}_{i_3} K^{j_4}_{i_4} \right) \times \cdots \times \left(\frac{1}{2} \mathcal{R}^{j_{2p-1} j_{2p}}_{i_{2p-1} i_{2p}} - s^2 K^{j_{2p-1}}_{i_{2p-1}} K^{j_{2p}}_{i_{2p}} \right) \Bigg) . \quad (2.46)$$

Notice that in the second term of the expression (2.45), polynomial coefficients derived from the parametric integrals have the same form as those from π^{ij} in Eq.(2.42). In fact, it is straighforward to show that this piece equals $\pi^{ij} h'_{ij}$ if one keeps in mind that $h'_{ij} = 2NK_{ij}$. Consequently, in regard to the Hamiltonian density, the Legendre transform of the Lagrangian (2.45) $\mathcal{H} = \pi^{ij} h'_{ij} - \mathcal{L}^{(1)}$ is simply

$$\mathcal{H} = -N \sum_{p=0}^{\left\lfloor \frac{D-1}{2} \right\rfloor} \alpha_p \mathcal{H}_p . \quad (2.47)$$

More generally, employing Eq.(2.27) to include also the shift function, we get

$$\mathcal{H} = N\mathscr{H} + N^i \mathscr{H}_i \,, \tag{2.48}$$

where the constrains $\mathscr{H}$ and $\mathscr{H}_i$ take the form

$$\begin{aligned} \mathscr{H} &= -\sum_{p=0}^{\lfloor \frac{D-1}{2} \rfloor} \alpha_p \mathcal{H}_p \,, \\ \mathscr{H}_i &= -2\bar{\nabla}_j \pi_i^j \,. \end{aligned} \tag{2.49}$$

The above constrains, in consistence with General Relativity, are related to the equations of motion as $\mathcal{E}_r^r = \mathscr{H}/2\sqrt{|h|}$ and $\mathcal{E}_i^r = \mathscr{H}_i/2N\sqrt{|h|}$. In fact, the Hamiltonian dynamics obtained from the variables N, N_i and h_{ij} [68] match the Lagrangian formalism following the correspondence

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial N} &= 0 \;\;\Leftrightarrow\;\; \mathcal{E}_r^r = 0 \,, \\ \frac{\partial \mathcal{H}}{\partial N^i} &= 0 \;\;\Leftrightarrow\;\; \mathcal{E}_i^r = 0 \,, \\ \frac{\partial \mathcal{H}}{\partial h_{ij}} - \partial_k \frac{\partial \mathcal{H}}{\partial \big(\partial_k h_{ij}\big)} + \partial_l \partial_k \frac{\partial \mathcal{H}}{\partial \big(\partial_l \partial_k h_{ij}\big)} &= -\big(\pi^{ij}\big)' \Leftrightarrow \mathcal{E}^{ij} = 0 \,. \end{aligned} \tag{2.50}$$

The expressions found here match those derived in Ref.[38]. The authors obtained the expression

$$\pounds_p = N\sqrt{|h|} \sum_{i=0}^{p} \tilde{C}_{i(p)} \delta^{[i_1 \cdots i_{2p}]}_{[j_1 \cdots j_{2p}]} R^{j_1 j_2}_{i_1 i_2} \cdots R^{j_{2i-1} j_{2i}}_{i_{2i-1} i_{2i}} K^{j_{2i+1}}_{j_{2i+1}} \cdots K^{j_{2p}}_{j_{2p}} \,, \tag{2.51}$$

with coefficients

$$\tilde{C}_{i(p)} = \frac{(-4)^{p-i}}{2\, i! [2(p-i) - 1]!!} \,. \tag{2.52}$$

In order to visualize the connection, consider the schematic representation $x = R^{ij}_{kl}$ and $y = K^i_j$ for $\mathcal{L}_p^{(1)}$ in Eq.(2.45) to rewrite it as

$$\mathcal{L}_p^{(1)} = \frac{x^p}{2^p} + 2p\epsilon \int_0^1 ds\, y^2 \left(\frac{1}{2}x + \big(1 - s^2\big)\epsilon y^2\right)^{p-1} = \sum_{i=0}^{p} C_{i(p)} x^i y^{2p-2i} \,, \tag{2.53}$$

or, reversing the labels x and y,

$$\mathcal{L}_p^{(1)} = N\sqrt{|h|} \sum_{i=0}^{p} C_{i(p)} \delta^{[i_1 \cdots i_{2p}]}_{[j_1 \cdots j_{2p}]} R^{j_1 j_2}_{i_1 i_2} \cdots R^{j_{2i-1} j_{2i}}_{i_{2i-1} i_{2i}} K^{j_{2i+1}}_{j_{2i+1}} \cdots K^{j_{2p}}_{j_{2p}} \,, \tag{2.54}$$

where

$$C_{i(p)} = \frac{p!2^{p-2i}\epsilon^{p-i}}{i!(2(p-i)-1)!!}\,. \tag{2.55}$$

The expressions $\pounds_p$ and $\mathcal{L}_p^{(1)}$ are in agreement up to an overall factor $p!/2^{p-1}$, due to different conventions.

2.4.1 An alternative method: Integration of the canonical momentum

One may argue that, since the integrand of variations of the Dirichlet action (2.38) have the form $\pi^{ij}\delta h_{ij}$, according to Classical Field Theory one expects a first-order Lagrangian form. In other words, by identifying the tensor density (2.41) as the associated canonical momentum for a certain $\mathcal{L}^{(1)}$, we must solve the partial differential equation

$$\frac{\partial\mathcal{L}^{(1)}}{\partial(h'_{ij})} = \pi^{ij}\,. \tag{2.56}$$

For simplicity, we may focusing in the p-th contribution. The radial derivative of the boundary metric obeys $h'_{ij} = -2NK_{ij}$ and integration of the above equality as a polynomial of the extrinsic curvature yields

$$\mathcal{L}_p^{(1)} = N\kappa(h_{ij},\partial_k h_{ij},\partial_k\partial_l h_{ij}) + 2pN\sqrt{-h}\int_0^1 ds(1-s)\delta^{[i_1\cdots i_{2p}]}_{[j_1\cdots j_{2p}]}K^{j_1}_{i_1}\times$$
$$\times K^{j_2}_{i_2}\left(\frac{1}{2}\bar{R}^{j_3j_4}_{i_3i_4} - s^2\epsilon K^{j_3}_{i_3}K^{j_4}_{i_4}\right)\times\cdots\times\left(\frac{1}{2}\bar{R}^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}} - s^2\epsilon K^{j_{2p-1}}_{i_{2p-1}}K^{j_{2p}}_{i_{2p}}\right)\,. \tag{2.57}$$

where $\kappa(h_{ij},\partial_k h_{ij},\partial_k\partial_l h_{ij})$ is a function that does not depend on normal derivatives of the induced metric. In general, κ can not be determined in this setup. However, in view of the Gauss-Codazzi relations, the only intrinsic quantity coming from a $(d+1)$ decomposition of the Riemann tensor is $\bar{R}^{ij}_{kl}$. In other words, κ can only be the p-th Lovelock density (2.35) but computed using the induced metric, i.e. $r = \bar{\mathcal{L}}^{(p)}$ with

$$\bar{\mathcal{L}}^{(p)} = \frac{1}{2^p}\sqrt{|h|}\delta^{[i_1\cdots i_{2p}]}_{[j_1\cdots j_{2p}]}\bar{R}^{j_1j_2}_{i_1i_2}\cdots\bar{R}^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}}\,. \tag{2.58}$$

However, this argument is far from being a definite proof. Fortunately, the bulkanization procedure give the exact form of those terms and strengthens the conclusions.

2.4.2 Example: Einstein-Gauss-Bonnet gravity

As an example, let us consider the simplest case of Lovelock gravity beyond Einstein gravity, the Einstein-Gauss-Bonnet action

$$I_{\rm EGB}[g] = \frac{1}{16\pi G}\int\limits_{\mathcal{M}} d^Dx\sqrt{-g}\Big(R - 2\Lambda + \frac{\alpha_2}{4}\delta^{[\mu_1\cdots\mu_4]}_{[\nu_1\cdots\nu_4]}R^{\nu_1\nu_2}_{\mu_1\mu_2}R^{\nu_3\nu_4}_{\mu_3\mu_4}\Big)\,, \tag{2.59}$$

which corresponds to the set $\{\alpha_0 = -2\Lambda, \alpha_1 = 1, \alpha_2 = \alpha_2\}$ in the action (2.38). This case has been studied in, e.g, Refs. [69, 70]. After the addition of the boundary terms needed to have a well-posed Dirichlet problem, the action takes the form

$$\tilde{I}_{\rm EGB} = I_{\rm EGB} - \frac{1}{8\pi G}\int\limits_{\partial\mathcal{M}} d^dx\sqrt{-h}\left(K + \alpha_2\delta^{[i_1i_2i_3]}_{[j_1j_2j_3]}K^{j_1}_{i_1}\left(\mathcal{R}^{j_2j_3}_{i_2i_3} - \frac{2}{3}K^{j_2}_{i_2}K^{j_3}_{i_3}\right)\right). \tag{2.60}$$

In Gaussian coordinates, the variation of (2.60) adopts the form

$$\delta\tilde{I}_{\rm EGB} = \frac{1}{16\pi G}\left(-\int\limits_{\mathcal{M}} d^Dx\sqrt{-g}H^{\mu\nu}\delta g_{\mu\nu} + \int\limits_{\partial\mathcal{M}} d^dx\sqrt{-h}\pi^{ij}_{\rm EGB}\delta h_{ij}\right), \tag{2.61}$$

where the equations of motion are

$$H^{\mu}_{\nu} = \Lambda\delta^{\mu}_{\nu} - \frac{1}{4}\delta^{[\mu\,\mu_1\mu_2]}_{[\nu\,\nu_1\nu_2]}R^{\nu_1\nu_2}_{\mu_1\mu_2} - \frac{1}{8}\delta^{[\mu\,\mu_1\mu_2\mu_3\mu_4]}_{[\nu\,\nu_1\nu_2\nu_3\nu_4]}R^{\nu_1\nu_2}_{\mu_1\mu_2}R^{\nu_3\nu_4}_{\mu_3\mu_4}\,, \tag{2.62}$$

and the canonical momentum is given by

$$\pi^{ij}_{\rm EGB} = -\sqrt{-h}h^{ik}\left(\delta^{[jj_1]}_{[ki_1]}K^{i_1}_{j_1} + \delta^{[jj_1j_2j_3]}_{[ki_1i_2i_3]}K^{i_1}_{j_1}\left(\mathcal{R}^{i_2i_3}_{j_2j_3} - \frac{2}{3}K^{i_2}_{j_2}K^{i_3}_{j_3}\right)\right). \tag{2.63}$$

Observe that the first and second terms in Eq.(2.62) are the cosmological constant and Einstein tensor respectively. Also, the first term in Eq.(2.63) matches the canonical momentum obtained for General Relativity. Then, we proceed to compute the first-order form for the Lagrangian associated to the action (2.60). From Appendix

C, the bulkanization procedure for the Einstein-Gauss-Bonnet case leads to the relation

$$\tilde{I}_{\mathrm{EGB}} = \int\limits_{\mathcal{M}} d^D x\sqrt{-g}\Bigg(-2\Lambda + \mathcal{R} - K^2 + K^i_j K^j_i + \frac{\alpha_2}{4}\delta^{[i_1\cdots i_4]}_{[j_1\cdots j_4]} R^{j_1j_2}_{i_1i_2} R^{j_3j_4}_{i_3i_4} \\ +2\alpha_2 \delta^{[i_1\cdots i_4]}_{[j_1\cdots j_4]} K^{j_1}_{i_1} K^{j_2}_{i_2}\left(\mathcal{R}^{j_3j_4}_{i_3i_4} - \frac{2}{3}K^{j_3}_{i_3}K^{j_4}_{i_4}\right)\Bigg). \tag{2.64}$$

Then, the first-order Lagrangian is given by

$$\mathcal{L}^{(1)}_{\mathrm{EGB}} = N\sqrt{-h}\Bigg(-2\Lambda + \mathcal{R} - K^2 + K^i_j K^j_i + \frac{\alpha_2}{4}\delta^{[i_1\cdots i_4]}_{[j_1\cdots j_4]} R^{j_1j_2}_{i_1i_2} R^{j_3j_4}_{i_3i_4} + \\ 2\alpha_2 \delta^{[i_1\cdots i_4]}_{[j_1\cdots j_4]} K^{j_1}_{i_1} K^{j_2}_{i_2}\left(\mathcal{R}^{j_3j_4}_{i_3i_4} - \frac{2}{3}K^{j_3}_{i_3}K^{j_4}_{i_4}\right)\Bigg). \tag{2.65}$$

The terms $\mathcal{R}-K^2+K^i_jK^j_i$, that corresponds to the Einstein part, can also be written as

$$\mathcal{R} - K^2 + K^i_j K^j_i = \frac{1}{2}\delta^{[i_1i_2]}_{[j_1j_2]} R^{j_1j_2}_{i_1i_2} + 2\delta^{[i_1i_2]}_{[j_1j_2]} K^{j_1}_{i_1} K^{j_2}_{i_2} \tag{2.66}$$

in consistence with Eq.(2.35). In fact, written in this way, one can directly recognize the terms proportional to H^r_r, while the rest is just $\pi^{ij}_{\mathrm{EGB}} h'_{ij}$. Thus, the Hamiltonian density $\mathcal{H}_{\mathrm{EGB}} = \pi^{ij}_{\mathrm{EGB}} h'_{ij} - \mathcal{L}^{(1)}_{\mathrm{EGB}}$ is given by

$$\mathcal{H}_{\mathrm{EGB}} = -N\sqrt{-h}\left(-2\Lambda + \frac{1}{2}\delta^{[i_1i_2]}_{[j_1j_2]} R^{j_1j_2}_{i_1i_2} + \frac{\alpha_2}{4}\delta^{[i_1\cdots i_4]}_{[j_1\cdots j_4]} R^{j_1j_2}_{i_1i_2} R^{j_3j_4}_{i_3i_4}\right). \tag{2.67}$$

The inclusion of the shift function, to recover the full ADM information, leads to the density functional

$$\mathcal{H}_{\mathrm{EGB}} = N\mathscr{H}_{\mathrm{EGB}} + N_i \mathscr{H}^i_{\mathrm{EGB}}\,, \tag{2.68}$$

where the constrains $\mathscr{H}_{\mathrm{EGB}}$ and $\mathscr{H}^i_{\mathrm{EGB}}$ satisfy

$$\mathscr{H}_{\mathrm{EGB}} = -\sqrt{-h}\left(-2\Lambda + \frac{1}{2}\delta^{[i_1i_2]}_{[j_1j_2]} R^{j_1j_2}_{i_1i_2} + \frac{\alpha_2}{4}\delta^{[i_1\cdots i_4]}_{[j_1\cdots j_4]} R^{j_1j_2}_{i_1i_2} R^{j_3j_4}_{i_3i_4}\right), \\ \mathscr{H}^i_{\mathrm{EGB}} = -2\bar{\nabla}_j \pi^{ij}_{\mathrm{EGB}}\,. \tag{2.69}$$

Chapter 3

Thin shell dynamics in Lovelock gravity

Thin shell theory focus on simplifying the dynamics of bodies whose thickness magnitude is much lower than the rest of its length scales. Its objective is to lower the dimension and simplify the resulting equations. For this purpose, the shell quantities are projected and expanded in powers of the thickness. Depending of how thin the sample objects are, one may preserve as many terms as needed that represent a reasonable approximation for the analyzed problem. In chapter 2, two simple examples were worked out that display the guideline logic for the case presented here.

However, junction conditions are not bounded by shells. In principle, they describe the behaviour of physical fields across surfaces where the matter density present discontinuities. For example, the case of Snyder-Oppenheimer collapse [51], where the discontinuity is exhibited with a Heaviside step function centered at the boundary of a star.

Junction conditions in the context of gravity was first explored by Israel [54]. He showed that integration of the equations of motion along an infinitesimal normal section enclosing the shell, predicts a jump in a precise combination of extrinsic and intrinsic quantities proportional to the localized matter. Such quantities, along with the equations of motion, were previously projected to the shell frame. The same conclusion were then obtained later for Einstein-Gauss-Bonnet (EGB) gravity [56–58, 72, 73].

A broadly explored application of matching conditions is braneworld scenarios [74], where the setup was a four-dimensional universe embedded in a five-dimensional spacetime as a thin brane. Later, modifications to Einstein gravity were also studied [61, 75]. In particular, it was shown that different types of junction conditions can be obtained depending on how differentiable the metric is asked to be [49].

It was shown that the junction conditions are also a consequence of the variational principle. However, Einstein and Lovelock gravity without boundary terms added depend on second-order derivatives. Fortunately, in the above chapter, a first-order formalism was introduced, that will be the basis to follow the steps of Classical Mechanics. This is also the reason why Lovelock gravity is the subject of study. Consequently, it is clear that the canonical momentum plays a role in matching conditions.

As will be shown later, the variational principle and equations of motion written in adapted coordinates on the shell, provide the shell dynamics in Lovelock gravity in the form of a jump in the canonical momentum across it, as explained by Israel in Ref.[54].

3.1 Junction conditions in Classical Mechanics

To show how the junction conditions work, we will check two simple examples of phase/configuration spaces which constain two solutions at the same time. The first example is a string with a mass fixed at some location and the second example is a derivation of interface conditions in Electromagnetism for a certain charged hypersurface. This chapter is structured to commit a logic guideline for the work done for gravity.

3.1.1 The guitar string with a point mass attached

The string is a continuous object and in order to write its Lagrangian we have to study the dynamics of its an infinitesimal piece. We will label with x the direction along the length of the string in the rest frame and y the perpendicular direction of the displacement, such that $y = y(x,t)$. For a section Δx, the kinetic energy is just one half of the square of the time derivative of the displacement multiplied by the mass of the infinitesimal piece, that we will call λ. The potential energy can be computed subtracting the rest length from the current string length and multiplying by the string tension T to obtain [65]

$$L_{\Delta x} = \frac{1}{2}\lambda \dot{y}^2 - T\left(\sqrt{\Delta x^2 + \Delta y^2} - \Delta x\right) . \tag{3.1}$$

We sum over its whole length L all contributions for an infinitesimal Δx to obtain, for small oscillations, the action [cite]

$$I = \frac{1}{2}\int_0^L \int_{t_1}^{t_2} dx\, dt\, \left(\lambda \dot{y}^2 - T y'^2\right) \tag{3.2}$$

where a prime stands for derivatives with respect to x and now λ is the linear density of the string. Variation of the above action gives

$$\delta I = \int_0^L \int_{t_i}^{t_f} dx\, dt\, \left(T y'' - \lambda \ddot{y}\right) \delta y + \int_0^L dx\, \left(\lambda \dot{y} \delta y\right) \Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} dt\, \left(T y' \delta y\right) \Big|_0^L ,$$

that requires not only two initial conditions, but also additional data regarding the string at all times at its ending points. Since the problem at hand is a guitar string, we know that for all times $y(0) = y(L) = 0$ as the borders of the string are fixed. This action principle is solved with Dirichlet boundary conditions at both boundaries and leads to the equation of motion for a wave moving at velocity $v = \sqrt{T/\lambda}$ [65, 66]

$$\ddot{y} = \frac{T}{\lambda} y'' . \tag{3.3}$$

Until now, this is a clean guitar string. Next, we introduce a point mass at the middle of the string through the redefinition

$$\lambda \to \lambda(x) = \lambda_0 + m\delta\left(x - \frac{L}{2}\right) \tag{3.4}$$

with mass density of the string λ_0 constant and m the mass of the point particle. Conveniently, this change do not spoil the validity of the above equations, the calculations are the same regardless of whether λ depends on x or not. However, we must now pay special attention to the point $x = L/2$. The variation of the action for the new Lagrangian is just

$$\delta I = \int_0^L \int_{t_i}^{t_f} dx\, dt\, \left(Ty'' - \lambda(x)\ddot{y}\right)\delta y + \int_0^L dx\, \left(\lambda(x)\dot{y}\delta y\right)\Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} dt\, \left(Ty'\delta y\right)\Big|_0^L .$$

At this point, we have two equivalent ways to describe the dynamics. The first, to treat the delta function as a bulk source for the equations of motion. The second, to transfer the source to the boundary, evaluating the delta function. In the former, the dynamics are just described by

$$\lambda_0\ddot{y} - Ty'' = -m\delta\left(x - \frac{L}{2}\right)\ddot{y}\,, \tag{3.5}$$

for all $x \in [0, L]$. The junction conditions are recovered integrating the above equation in an infinitesimally small interval containing the point mass, i.e,

$$\begin{aligned} \lambda_0\ddot{y}^2 - Ty'' &= -m\delta\left(x - \frac{L}{2}\right)\ddot{y} \Big/ \lim_{\varepsilon\to 0}\int_{\frac{L}{2}-\varepsilon}^{\frac{L}{2}+\varepsilon} dx\,(\cdot) \\ [\![y']\!] &= \frac{m}{T}\ddot{y}\,. \end{aligned} \tag{3.6}$$

where $[\![f]\!] = f_+ - f_-$ stands for the difference of f between the left and right sections. On the other hand, in the latter, we can integrate the delta function to the boundary in the vicinity of the point mass. To do so, we split the domain of the integral (3.2) for x in the three intervals $[0, L/2-\varepsilon[$, $[L/2-\varepsilon, L/2+\varepsilon]$ and $]L/2+\varepsilon, L]$ for small positive ε. By doing so, we are introducing a new shared boundary at $x = L/2$ for the left and right intervals $[0, L/2-\varepsilon[$ and $]L/2+\varepsilon, L]$ respectively. The Lagrangian dynamics are equivalent for both intervals to the clean guitar string dynamics except for the new boundary. The variational principle at the point particle takes the form

$$\delta I = -\int_{t_i}^{t_f} dt \left(-T\, [\![y']\!] + m\ddot{y}\right) \delta y \,,$$

where in contrast, the equations of motion are valid for $x \in [0, L] - \{L/2\}$ given by Eq.(3.3). In the above result, we also assume that the string is continuous, such that $y(x)$ does not jump between regions. Furthermore, because we want δy to fluctuate with the system at the point mass –otherwise the particle will be a fixed point in the geometry like the string edges–, the new boundary at $x = L/2$ does not apply for the Dirichlet problem. To fulfil the variational principle, the factor next to δy have to vanish, leading to the junction condition (3.6). Notice that the quantity that jumps

$$p_x = \frac{\partial L}{\partial y'} = T y' \tag{3.7}$$

is nothing but the canonical momentum associated to y for an evolution onward x. This second procedure may seem more tedious, however, it is useful when the terms in the equations of motion are complicated to work out. We can analyze the solution at some limit cases to understand the information from the junction conditions. For example, the difference between the left and right velocities scales with the mass. This is expected as a heavier mass will influence further the movement of the string. On the other hand, the mass particle influence is inversely proportional to the tension, that is aligned with home observations.

3.1.2 The charged shell in Electrodynamics

For Electrodynamics, the Lagrangian that recovers Maxwell equations is

$$I = \int\limits_{\mathcal{M}} d^4x \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu J^\mu \right) \,, \tag{3.8}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell tensor constructed from the electromagnetic 4-potential A_μ. The current density J^μ is a matter source. In our case, we will analyze localized matter, such that the current density is proportional to a delta function. For simplicity we are working in flat space with the Minkowski metric $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. Variations of the above action functional leads to

$$\delta I = \int\limits_{\mathcal{M}} d^4x \left(-\partial_\mu F^{\mu\nu} + J^\nu\right) \delta A_\nu + \int\limits_{\partial\mathcal{M}} d^3x \big(n_\mu F^{\mu\nu} \delta A_\nu\big) \,, \tag{3.9}$$

where n^μ is the unit normal to the hypersurface $\partial\mathcal{M}$. Once again, the variational principle is fulfilled for Dirichlet boundary conditions $\delta A_\nu|_{\partial\mathcal{M}} = 0$, whose bulk contribution implies the Maxwell equations $\partial_\mu F^{\mu\nu} = J^\nu$. Now, in a similar fashion to the previous example of a string, we place the matter content in a domain wall through the current $J^\mu = j^\mu \delta(n)$, where n is the normal to the shell. Once more, there are two ways to recover the junction conditions, and they differ in where we keep the matter source. Firstly, if J^μ is kept at the bulk, the matching conditions can be derived integrating the equations of motion in a Gaussian pillbox enclosing the shell. On the other hand, we can integrate the junction conditions to the boundary. The shell is the matter source and at the same time it is a boundary for the inner and outer spaces. Therefore, the interface condition can be read from the boundary term next to δA_ν to get

$$n_\mu \left[\!\left[F^{\mu\nu} \right]\!\right] = j^\nu \,. \tag{3.10}$$

In search of a dynamic shell, it is natural to keep δA_ν different from zero. Otherwise, the shell will be static, in other words, a fixed hypersurface in the resulting geometry. The full spacetime is also continuous, which secures continuity for any continuous function of the coordinates.

As example, consider the case of a charged spherical shell with charge density σ, which will be useful to tackle the problem discussed in Chapter 3. In this setup, the normal vector is along the radial coordinate r, such that $n_\mu = \delta^r_\mu$, which leads to the condition

$$E_{outer} - E_{inner} = \sigma \,, \tag{3.11}$$

where E stands for the radial component of the electric field, given by $\partial_r \phi$, with ϕ being the electric potential.

Another example is given by a charged plate in a flat spacetime described by coordinates (x, y, z, t) and $n_\mu = \delta^z_\mu$ (normal to a constant x hypersurface). If now σ represents the charge density of the plate located at $z = 0$, the above junction conditions reproduce the relation

$$E_+ - E_- = \sigma \,, \tag{3.12}$$

where $E_\pm$ represent z component of the left and right (to the plate) portions of the electric field, respectively, of the full spacetime. Next, we focus on the basic concepts necessary to work in the framework of General Relativity.

3.2 Two procedures to parametrize the shell position

First: A succinctly review of how to parametrize thin shells in the context of gravity, as done in e.g. Refs.[62, 63], is presented.

Imagine a manifold $\mathcal{M}$ separated in two empty regions by a thin spherical shell located at a certain, not fixed, radius. By virtue of Birkhoff theorem for the action (2.23) after assuming spherical symmetry, the inner ($-$) and outer ($+$) regions are described by the Schwarzchild metric

$$ds^2_\pm = g^\pm_{\mu\nu} dx^\mu dx^\nu = -f^2_\pm(r) dt^2_\pm + \frac{dr^2}{f^2_\pm(r)} + r^2 d\Omega^2 \,, \tag{3.13}$$

where $t_\pm$ are the outer and inner time coordinates and $d\Omega^2 = \omega_{mn} dx^m dx^n$ is the metric of an unitary S_{D-2}. In order to describe the shell, one analyze an observer standing on it, whose dynamics should be outlined by the metric

$$ds^2 = h_{ab} dy^a dy^b = -d\tau^2 + R^2(\tau) d\Omega^2 \,, \tag{3.14}$$

where $y^a = \{\tau, \theta_1, \ldots, \theta_{D-2}\}$ label the shell proper coordinates. In the metric above, the proper length was set such that the origin is exactly at the shell position and thus constant along the motion. Consequently, the position radius R becomes a functional purely of τ.

From now on, the sublabel $\pm$ is omitted since the computations are equal for inner and outer regions. After comparing Eq.(3.14) with Eq.(3.13), the condition that determines $R(\tau)$ is

$$f^2(R)\dot{t}^2 - \frac{\dot{R}^2}{f^2(R)} = 1\,, \tag{3.15}$$

where a dot stands for ∂_τ. Since the spacetime variables are $x^\alpha = (r, t, \theta_1, \cdots, \theta_{D-2})$, the tangent vector to the shell surface, provided $r = R(\tau)$ and $t = t(\tau)$, is simply

$$u^\alpha = \frac{dx^\alpha}{d\tau} = (\dot{R}, \dot{t}, \vec{0}) = \left(\dot{R}, \frac{\gamma}{f^2(R)}, \vec{0}\right) \tag{3.16}$$

where we defined the (modified) relativistic factor

$$\gamma = f^2(R)\dot{t} = \sqrt{\dot{R}^2 + f^2(R)}\,. \tag{3.17}$$

The normal can be computed directly from the facts $n^\alpha u_\alpha = 0$, $n^\alpha n_\alpha = \pm 1$ and that it lies in the (r, t) plane. The outcome is

$$n^\alpha = \left(\gamma, \frac{\dot{R}}{f^2(R)}, \vec{0}\right)\,. \tag{3.18}$$

Second: The authors in Ref.[77] show a more general approach employing the coordinate transformation for the (r, t) plane

$$r = r\left(\lambda, \tau\right) \ \text{and} \ t = t\left(\lambda, \tau\right)\,, \tag{3.19}$$

that in contrast with the above case, leaves λ as a normal coordinate. The metric (3.13) in the new variables is then

$$\begin{aligned} ds^2 = g_{AB}dy^A dy^B \;\; &= \;\; -\left(f^2(r)\dot{t}^2 - \frac{\dot{r}^2}{f^2(r)}\right)d\tau^2 + 2\left(\frac{\dot{r}\partial_\lambda r}{f^2(r)} - f^2(r)\dot{t}\partial_\lambda t\right)d\tau d\lambda \\ &\quad + \left(\frac{\left(\partial_\lambda r\right)^2}{f^2(r)} - f^2(r)\left(\partial_\lambda t\right)^2\right)d\lambda^2 + r^2 d\Omega^2\,, \end{aligned} \tag{3.20}$$

where $y^A = (\lambda, \tau, \theta_1, \ldots, \theta_{D-2})$. Up to here, this procedure is just a change of coordinates and therefore is valid in the whole D-dimensional manifold. In a patch that enclose the shell, comparison of the above and shell expected metric

$$ds^2 = d\lambda^2 - d\tau^2 + r^2\left(\lambda, \tau\right) d\Omega^2 \,. \tag{3.21}$$

derive the conditions

$$\begin{aligned} f^2(r)\dot{t}^2 - \frac{\dot{r}^2}{f^2(r)} &= 1\,, \\ \frac{\dot{r}\partial_\lambda r}{f^2(r)} - f^2(r)\dot{t}\partial_\lambda t &= 0\,, \\ \frac{\left(\partial_\lambda r\right)^2}{f^2(r)} - f^2(r)\left(\partial_\lambda t\right)^2 &= 1\,. \end{aligned} \tag{3.22}$$

Notice that in opposition to the previous example, there are two new constraints, which are necessary to fix the normal derivatives of r and t. In order to obtain the extrinsic and intrinsic curvature tensors in the new set of variables y^A, needed later, the associated Jacobian matrix

$$e^\mu_A = \frac{\partial x^\mu}{\partial y^A}\,, \tag{3.23}$$

is needed. The normal vector and the four-velocity, perpendicular and tangent to the shell respectively, are given by

$$e^\mu_\lambda = n^\mu = \frac{dx^\mu}{d\lambda} \quad , \quad e^\mu_\tau = u^\mu = \frac{dx^\mu}{d\tau}\,, \tag{3.24}$$

which allows to write the d-dimensional induced metric as

$$h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b \,. \tag{3.25}$$

With this data, the extrinsic curvature is calculated using the formula

$$K_{ab} = e^\alpha_a e^\beta_b \nabla_\alpha n_\beta \,. \tag{3.26}$$

to obtain the expressions

$$K_{\tau\tau} = \gamma' \quad , \quad K_{nm} = -\gamma r \omega_{mn} \,. \tag{3.27}$$

In addition, the non-vanishing components of the intrinsic curvatures for the induced metric are

$$\mathcal{R}^{\tau m}_{\tau n} = \frac{\ddot{R}}{R}\delta^m_n \quad , \quad \mathcal{R}^{nm}_{pq} = \frac{1+\dot{R}^2}{R^2}\delta^{[nm]}_{[pq]} \,. \tag{3.28}$$

In order to characterize the matter content, the energy-momentum tensor have the form

$$T^{AB} = S^{ab}\delta_a^A\delta_b^B\delta(\lambda)\,. \tag{3.29}$$

where S^{ab} is the d-dimensional energy-momentum tensor of the shell and $\delta(\lambda)$ secures that the mass is localized.

3.3 The junction conditions for Lovelock gravity from two points of view

3.3.1 The junction conditions from the variational principle

In this subsection, the goal is to show that junction conditions can be obtained by requiring a well-posed variational principle. Recall the Dirichlet action (2.38), whose variation in Gauss-normal coordinates produces Eq.(2.39). When the thin shell is added to the geometry, the spacetime is split in two regions with outer and inner metrics $g^+_{\mu\nu}$ and $g^-_{\mu\nu}$, respectively, sharing a boundary at $r = R$. Now, the coordinates (r, t) are useful to apply Dirichlet conditions at infinity but are not the most convenient near the shell. To prevent working with the D coordinates x^μ in the d-dimensional shell hypersurface, an option is to employ the set y^A, which naturally lead to the aforementioned hypersurface when $\lambda = 0$.

Thus, in a patch that covers the shell, we will use instead the adapted coordinates defined in (3.19) to obtain the bulk quantities as a function of the shell coordinates. Notice that due to the conditions (3.22), in a patch near the shell, the metric (3.20) adopts a Gauss-normal form with λ as normal coordinate. Consequently, the Lagrangian scalar, written in terms of the variables (τ, λ), lead to the expression

$$\delta I_{\text{Dir}} = \frac{1}{16\pi G}\int\limits_{\mathcal{M}} d^D y \partial_\lambda \big(\pi^{ab}\delta h_{ab}\big)\,. \tag{3.30}$$

This result requires writing a Dirichlet action for the coordinates y^A, which implies that a Gibbons-Hawking-York term is added at the shell. Consequently, the latter is function of the boundary coordinates y^a and not x^i which is the variables for the terms at radial infinity.

The boundary conditions at infinity are not relevant at finite radius. Therefore, we are left with the following contribution in a small integration domain along the normal direction,

$$\delta I_{Dir} = \frac{1}{16\pi G}\int\limits_{\partial\mathcal{M}} d^d y \big(\pi_+^{ab}\delta h_{ab}^+ - \pi_-^{ab}\delta h_{ab}^-\big)\,. \tag{3.31}$$

At this point, we relax the (Riemannian) condition of g being smooth and consider a metric at least continuous at the shell

$$h_{ab}^+\big|_{\partial\mathcal{M}} = h_{ab}^-\big|_{\partial\mathcal{M}} = h_{ab}\,, \tag{3.32}$$

where $h_{ab}^\pm$ portray the limits

$$h_{ab}^+\big|_{\partial\mathcal{M}} = \lim_{\varepsilon\to 0} h_{ab}\left(\varepsilon,\tau\right) \quad , \quad h_{ab}^-\big|_{\partial\mathcal{M}} = \lim_{\varepsilon\to 0} h_{ab}\left(-\varepsilon,\tau\right) \tag{3.33}$$

it is clear that $\delta h^+|_{\partial\mathcal{M}} = \delta h^-|_{\partial\mathcal{M}} = \delta h$. The energy momentum tensor (3.29) is a delta function located at the shell and its inclusion renders (3.31) to

$$\delta I_{\text{Dir}} + \delta I_{\text{matter}} = \frac{1}{16\pi G}\int\limits_{\partial\mathcal{M}} d^d y \big(\left[\!\left[\pi^{ab}\right]\!\right] - 8\pi G S^{ab}\sqrt{-h}\big)\delta h_{ab}\,. \tag{3.34}$$

Since δh_{ab} must be arbitrary, if one expect to obtain shell dynamics, the variational principle requires

$$\left[\!\left[\pi^{ab}\right]\!\right] = 8\pi G S^{ab}\sqrt{-h}, \tag{3.35}$$

where $[\![f]\!] = f_+ - f_-$ stand for the difference of the functional between the outer and inner region. Due to the assumption that h_{ij} is continuous, it is also possible to rewrite Eq.(3.35) as

$$\left[\!\left[\tilde{\pi}^{ab}\right]\!\right] = 8\pi G S^{ab}, \tag{3.36}$$

where the expression

$$\tilde{\pi}^{ab} = \frac{\pi^{ab}}{\sqrt{-h}} \tag{3.37}$$

is the tensorial part of the momentum tensor density π^{ab}.

3.3.2 The junction conditions from the equations of motion

In the variational principle, if instead of treating the shell at the boundary one keeps it at the bulk, the source couples to the equations of motion. In Ref.[54], Israel showed that the junction conditions can be obtained integrating the equations of motion projected to the shell along the normal direction. Near the shell, if the Lagrangian is provided with a suitable Gibbons-Hawking-York for it, then it is possible to attain a first-order form. Rewritten in the co-moving frame, the component transversal to the shell of the equations of motion is [68]

$$\frac{\partial \mathcal{L}^{(1)}}{\partial h_{ab}} - \partial_\lambda \left(\pi^{ab} \right) - \partial_c \left(\frac{\partial \mathcal{L}^{(1)}}{\partial_c h_{ab}} \right) + \partial_d \partial_c \left(\frac{\partial \mathcal{L}^{(1)}}{\partial_d \partial_c h_{ab}} \right) = 8\pi G T^{ab} \sqrt{-h}\,, \tag{3.38}$$

which is basically the Euler-Lagrange equation for $\mathcal{L}^{(1)}[h, K, \mathcal{R}]$.

Thus, in Lovelock gravity, the second order terms are packed in the derivative of the momentum tensor density. Since a delta distribution in present in the energy-momentum side, the second derivative terms are going to be the source of the delta function. The first derivative terms –the extrinsic curvatures– are Heaviside step functions. In this way, the metric can still be continuous (but not smooth) and the extrinsic curvatures are discontinuous but bounded functions. In other words, all terms except for the normal derivative of the canonical momentum in the l.h.s of Eq.(3.38) are bounded.

Therefore, Israel's procedure [54], starting from Eq.(3.38) and integrating in an interval $\lambda = \{-\varepsilon, \varepsilon\}$ leads, in the thin shell limit $\varepsilon \to 0$, to the expression Eq.(3.35).

3.4 Explicit expressions for spherically symmetric solutions

In order to make contact with some existing literature, the example of a thin pressureless dust shell is explored. The last is characterized by $S^{ab} = -\sigma u^a u^b$, where σ is the matter density and u^A the proper velocity. Since in this setup $u^a = \delta^a_\tau$, the study reduces to find $\pi^{\tau\tau}$ at $r = R(\tau)$ and simplify.

3.4.1 Review of the results in General Relativity

In the case of the Einstein-Hilbert action, the (τ, τ) component of the momentum tensor density is calculated using equations (2.42) and (3.27) to get

$$\tilde{\pi}^\tau_\tau = (D-2)\frac{\gamma}{R}\,. \tag{3.39}$$

From Eq.(3.35), some manipulations lead the junction conditions to

$$\gamma_+ - \gamma_- = -\frac{8\pi G\sigma}{D-2}\,, \tag{3.40}$$

in agreement with Refs.[62, 73]. Furthermore, recall that spherical symmetry in Einstein gravity implies the Schwarzschild spacetime which have

$$f^2(r) = 1 + \frac{r^2}{\ell^2} - \frac{16\pi GM}{(D-2)\Omega r^{D-3}}\,, \tag{3.41}$$

where Ω is the volume of a S_{D-2} From Eq.(3.39), after multiplying by $\gamma_+ + \gamma_-$ and replacing the expression for γ one gets

$$[\![M]\!] = m\frac{(\gamma_+ + \gamma_-)}{2}\,. \tag{3.42}$$

where $m = R^{D-2}\Omega\sigma$ is the proper mass of the shell

3.4.2 Junction conditions in Einstein-Gauss-Bonnet gravity

Einstein-Gauss-Bonnet (EGB) gravity is the simplest case of Lovelock gravity after General Relativity. It is described by the of first-order Lagrangian density $\mathcal{L}^{(1)}_{\text{EGB}} = \mathcal{L}^{(1)}_1 - 2\Lambda + \alpha\mathcal{L}^{(1)}_2$, where $\mathcal{L}^{(1)}_1 - 2\Lambda$ is the Einstein-Hilbert Lagrangian and α is the coupling constant of the Gauss-Bonnet term contribution $\mathcal{L}^{(1)}_2$.

The spherically symmetric solution is given by the Boulware-Deser metric [78], characterized by

$$f^2(r) = 1 + \frac{r^2}{2\tilde{\alpha}}\left[1 - \sqrt{1 - 4\tilde{\alpha}\left(\frac{1}{\ell^2} - \frac{16\pi GM}{(D-2)\Omega r^{D-1}}\right)}\right] \tag{3.43}$$

where for simplicity $\tilde{\alpha} = \alpha(D-3)(D-4)$ was defined. The computation of $\tilde{\pi}^\tau_\tau$ for EGB gravity yields

$$\tilde{\pi}^\tau_\tau \;=\; (D-2)\frac{\gamma}{R}\left(1 + 2\tilde{\alpha}\left(\frac{1 + \dot{R}^2 - \frac{1}{3}\gamma^2}{R^2}\right)\right), \tag{3.44}$$

in agreement with Refs.[72, 73]. The junction conditions in the form (3.36), after some factorization, give

$$\frac{(D-2)}{R}(\gamma_+ - \gamma_-)\left[1 + \frac{2\tilde{\alpha}}{R^2}\left(1 + \dot{R}^2 - \frac{1}{3}(\gamma_+^2 + \gamma_+\gamma_- + \gamma_-^2)\right)\right] = -8\pi G\sigma\,, \tag{3.45}$$

which in does not allow to easily solve for $\dot{R}$, nor to find a closed expression for $[\![M]\!]$, as it happens in the above case.

It is viable to find a simpler expression if some assumptions are implemented. In particular, consider the Lovelock Unique Vacuum (LUV) [76], that was also studied in Ref[64], defined by the coupling choice

$$\tilde{\alpha} = \frac{\ell^2}{4}\,. \tag{3.46}$$

Notice that this is equivalent to pick $\tilde{\alpha}$ such that all non-mass terms inside the radical in $f^2(r)$ cancel out. The jump in the canonical momentum can be rewritten for this case as

$$[\![\tilde{\pi}^{\tau}_{\tau}]\!] = \frac{(D-2)}{R}(\gamma_+ - \gamma_-)\left[\sqrt{\frac{16\pi G M_- \ell^2}{(D-2)\Omega R^{D-1}}} - \frac{2\tilde{\alpha}}{3R^2}\big(\gamma_+ - \gamma_-\big)\big(\gamma_+ + 2\gamma_-\big)\right]. \quad (3.47)$$

If the inner mass vanish, it is possible to achieve a form similar to the one obtained from GR. First, the above equation for $M_- = 0$ lead to

$$\frac{(D-2)\ell^2}{6R^3}\big(\gamma_+ - \gamma_-\big)^2\big(\gamma_+ + 2\gamma_-\big) = 8\pi G\sigma\,. \quad (3.48)$$

Then, multiplying both sides by $(\gamma_+ + \gamma_-)^2$ and replacing some $\gamma_\pm$ one gets

$$M_+ = \frac{3\big(\gamma_+ + \gamma_-\big)^2}{4\big(\gamma_+ + 2\gamma_-\big)}m\,. \quad (3.49)$$

Even though the form is somehow similar, the ponderation of γ_+ and γ_- is different, in contrast with General Relativity.

3.5 Junction conditions for Lovelock gravity

For Lovelock gravity, it is easier to analyze each term of degree p separately. In general, the momentum is

$$\tilde{\pi}^{\tau}_{\tau} = \sum_{p=0}^{\lfloor\frac{D-1}{2}\rfloor} \alpha_p \tilde{\pi}^{\tau}_{(p)\tau}\,, \quad (3.50)$$

where the p-th contribution is

$$\tilde{\pi}^{\tau}_{(p)\tau} \;=\; \frac{p(D-2)!}{(D-2p-1)!}\frac{\gamma}{R^{2p-1}}\int_0^1 dt\,\left(1+\dot{R}^2-\gamma^2t^2\right)^{p-1}. \quad (3.51)$$

This leads to the junction condition

$$\left[\!\!\left[\sum_{p=0}^{\lfloor\frac{D-1}{2}\rfloor} \alpha_p \frac{p(D-2)!}{(D-2p-1)!}\frac{\gamma}{R^{2p-1}}\int_0^1 dt\,\left(1+\dot{R}^2-\gamma^2t^2\right)^{p-1}\right]\!\!\right] = -8\pi G\sigma \quad (3.52)$$

that once again does not allow to solve explicitly for $\dot{R}$ or $[\![M]\!]$.

In general, it is not possible to go beyond this point as the solutions for $f^2(r)$ are not known. In the case of terms cubic and quartic in the curvature, in principle its possible to solve for $f^2(r)$ but the formulas are complicated to manipulate. For terms of degree higher than four in the curvature, the Abel-Ruffini theorem prevents explicit formulas. However, it is viable if the coupling constants are fixed. An example is the Lovelock Unique Vacuum theory mentioned above. In order to understand the above statements, let us first check how one should solve $f^2(r)$ in the general case.

In the Gaussian frame (A.10), the (r,r) component of the equations of motion (2.40) for the ansatz (3.13) takes the form (See also Ref.[76])

$$\begin{aligned} \mathcal{E}^r_r &= -\sum_{p=0}^{\lfloor\frac{D-1}{2}\rfloor} \alpha_p \mathcal{E}^{r(p)}_r \\ &= -\sum_{p=0}^{\lfloor\frac{D-1}{2}\rfloor} \frac{\alpha_p (D-2)!}{2(D-2p-1)! r^{D-2}} \left(r^{D-2p-1}\left(1-f^2(r)\right)^p\right)' \end{aligned} \tag{3.53}$$

Let us start with the simpler case, the Einstein-Hilbert Lagrangian, where the couplings are $\alpha_0 = -2\Lambda$ and $\alpha_1 = 1$. One gets for the first two terms in the summation (3.53) the results

$$\alpha_0 \mathcal{E}^{r(0)}_r = \Lambda \quad , \quad \alpha_1 \mathcal{E}^{r(1)}_r = -\frac{(D-2)}{2r^{D-2}} \left(r^{D-3}\left(1-f^2(r)\right)\right)' . \tag{3.54}$$

For pure gravity, that implies $\mathcal{E}^r_r = 0$, the differential equation becomes a total derivative equals to zero. The solution obtained for $f^2(r)$ is

$$f^2(r) = 1 - \frac{C}{r^{D-3}} - \frac{2\Lambda r^2}{(D-1)(D-2)} \tag{3.55}$$

where C is an integration constant. It is clear that this is no more than the Schwarzschild solution if one picks $C \propto M$. If now one includes the Gauss-Bonnet term with an arbitrary coupling, which corresponds to the Einstein-Gauss-Bonnet example, the problem reduces to solve

$$\frac{2\Lambda r^{D-2}}{(D-2)} - \left(r^{D-1}F(r)\right)' - \alpha_2 (D-3)(D-4)\left(r^{D-1}F(r)^2\right)' = 0 \tag{3.56}$$

where we made the change of variable

$$F(r) = \frac{1-f^2(r)}{r^2} . \tag{3.57}$$

Since the radial integration is free, the problem just reduces to solve a second order equation with $F(r)$ as the incognito. The solution for $f^2(r)$ will then have two branches, of which the Boulware-Deser solution (3.43) is the negative branch. The positive branch is not a black hole configuration and therefore it is not relevant here. The same game can be played when higher curvature corrections are added. In particular, as done in Ref.[76], one may ask the coefficients to be such that one can get rid of the summation using the binomial theorem. After integration, Eq.(3.13), for pure gravity, leads to the expression

$$\sum_{p=0}^{\lfloor\frac{D-1}{2}\rfloor} \frac{\alpha_p (D-2)!}{2(D-2p-1)!} F(r)^p = \frac{C}{r^{D-1}} \,. \tag{3.58}$$

We must keep in mind that we expect to recover Einstein gravity when the higher curvature terms are turned off. In other words, α_0 and α_1 are fixed. Assume we want the l.h.s of the above equation to reach a form proportional to $(F(r)+A)^k$, where A is some constant and k is the vacuum multiplicity. Thus, we rearrange Eq.(3.58) to construct the equality

$$\sum_{p=0}^{\lfloor\frac{D-1}{2}\rfloor} \frac{\alpha_p (D-1)!}{(D-2p-1)!} F(r)^p = \frac{C}{r^{D-1}} \,. \tag{3.59}$$

With the values of the first two coupling constants, we can deduce that the final form we want to obtain is

$$\frac{1}{\big(-2\Lambda\big)^{k-1}} \left(\frac{(D-1)(D-2)}{k} F(r) - 2\Lambda \right)^k = \frac{C}{r^{D-1}} \tag{3.60}$$

which fixes the Lovelock coefficients to the expression

$$\alpha_p = \frac{(D-2p-1)!}{(D-1)!} \binom{k}{p} \big(-2\Lambda\big) \left(\frac{(D-1)(D-2)}{-2\Lambda k} \right)^p \,. \tag{3.61}$$

The black hole solution for $f^2(r)$ for Eq.(3.60) is [76]

$$f^2(r) = 1 - \frac{2\Lambda k}{(D-1)(D-2)} r^2 - \sqrt[k]{\frac{C}{r^{D-2k-1}}} \tag{3.62}$$

where C is again, albeit different from previous cases, a constant related to the black hole mass.

With the convention proposed in Ref.[76]

$$\frac{1}{\ell_{\text{eff}}^2} = \frac{k}{\ell^2} = \frac{-2\Lambda k}{(D-1)(D-2)}\,, \tag{3.63}$$

where ℓ the AdS radius, the (τ,τ) component of the canonical momentum tensor in LUV theory is simply

$$\tilde{\pi}_\tau^\tau \;=\; \frac{(D-2)}{k\ell_{\text{eff}}^2}\sum_{p=1}^{k} p\,\ell_{\text{eff}}^{2p}\binom{k}{p}\frac{\gamma}{R^{2p-1}}\int_0^1 dt\,\left(1+\dot{R}^2-\gamma^2 t^2\right)^{p-1}\,. \tag{3.64}$$

With the help of the binomial theorem, the summation can be factorized to construct the expression

$$\tilde{\pi}_\tau^\tau = (D-2)\left(\frac{\ell_{\text{eff}}^2}{R^2}\right)^{k-1}\frac{\gamma}{R}\int_0^1 dt\,\left(1+\frac{R^2}{\ell_{\text{eff}}^2}+\dot{R}^2-\gamma^2 t^2\right)^{k-1}\,,$$

that leads to the junction condition

$$\left[\!\left[(D-2)\left(\frac{\ell_{\text{eff}}^2}{R^2}\right)^{k-1}\frac{\gamma}{R}\int_0^1 dt\,\left(1+\frac{R^2}{\ell_{\text{eff}}^2}+\dot{R}^2-\gamma^2 t^2\right)^{k-1}\right]\!\right] = -8\pi G\sigma\,. \tag{3.65}$$

The particular case of $M_- = 0$, explored for Einstein-Gauss-Bonnet, can also be reckoned for a general LUV theory of multiplicity k. It can be shown that $\gamma_+ - \gamma_-$ can be factorized k times in the l.h.s. of Eq.(3.65). However, the final result is not enlightening and is omitted.

Chapter 4

Conclusion

In this thesis, we proved that, for a Gaussian frame, each Lovelock density can be written as the sum of the derivative of its corresponding Myers' term plus a correction. In Lagrangian mechanics, for Lovelock gravity, the former is subtracted to achieve a well-posed Dirichlet problem and, therefore, the latter happens to be the final form of the Lagrangian. Then, it was shown that the form of the correction is proportional to the Hamiltonian constraint associated to the lapse function plus $\pi^{ij}h'_{ij}$. Furthermore, the corrections are a functional purely of the boundary metric h_{ij}, the extrinsic curvature K_{ij} and transversal derivatives of the boundary metric, which implies the reached Lagrangian has, at most, first-order derivatives with respect of the normal variable r. This fact allowed to exploit Classical Field Theory to directly compute the Hamiltonian of the system through a Legendre transformation. Also, it was shown that the dynamics of both descriptions are in agreement.

Even though some particular examples were computed, the results extend to any Lagrangian that can be written in first-order form[1]. Also, the boundary terms explored here are not the only choice for a well-posed Dirichlet problem. It is possible, in principle, to add any function of h_{ij} at the boundary without ruining the compatibility. The expressions presented here are the minimal terms needed when no other metrics are included. When a background metric is included in the action,

[1]For example, the axionic action in Ref.[81]

it is possible to define a different approach of the Dirichlet problem [70].

There are some constrains to mind for the couplings in the Lovelock action. It is probable that, for particular choices of $\{\lambda_p\}$, some of the components of the metric solution cannot be fully determined by the field equations [71]. For example, static spherically symmetric ansatz may remain arbitrary if the action has non-unique degenerate vacuum.

Both the variational principle and integration of the equations of motion lead to the same result. From Classical Mechanics, the variation of a Lagrangian $L(q,\dot{q})$ bring, by virtue of the chain rule, a term $p\delta\dot{q}$ –where $p=\partial L/\partial\dot{q}$–. This last is the source of both the term $p\delta q$ at the boundary along with $\dot{p}\delta q$ at the bulk. However, in the case of matching conditions for gravity, this link requires to correctly identify the normal direction and project all quantities to an appropriate coordinate system.

In regard to the explicit results for spherically symmetric solutions, Einstein gravity sees a jump between the inner and outer black hole mass proportional to the mass of the shell [54, 62]. However, for higher curvature Lagrangians the shell dynamics developed nonlinear combinations of the γ factors, in contrast with Ref.[64]. This mismatch can be attributed to the form of the equations of motion used in Ref.[64], that lost track of the canonical variables of the problem. However, it is not entirely unexpected to obtain correction. If one thinks about the example of the guitar string, the influence of the mass, according to the junction conditions, is inversely proportional to the tension. In other words, for high tension, the mass influences less and less the movement. In gravity, the cosmological constant can be understood as a background pressure. Therefore, since the influence is manifested through the background curvature, one may expect more influence if higher-order curvature terms are added. This property may be interesting in the study corrections for thermalization in the AdS/CFT context [31], as done in e.g. Ref.[77]. Here, the authors derive the shell equations in the AdS side with the objective of studying the evolution of the entanglement entropy in CFT side. An interesting problem is explored the implications of adding Lovelock terms in the gravity side.

In Chapter 3, it is shown that the matching conditions involve a jump of the canonical momentum. Brown and York exhibited that a Lagrangian compatible with Dirichlet boundary conditions has a conserved current proportional to the momentum tensor [80]. Their definition of conserved change is

$$\begin{aligned} Q_\xi &= -\frac{1}{16\pi G}\int_\Omega d^{D-2}x\sqrt{\sigma}\left(u_i\left(\pi^i_j-\pi^{[0]i}_j\right)\xi^j\right) \\ &= -\frac{1}{16\pi G}\int_\Omega d^{D-2}x\sqrt{\sigma}\left(u_i\pi^i_j\xi^j\right)-Q^{[0]}_\xi \end{aligned} \tag{4.1}$$

where u_i is the normal of the second foliation, $\sqrt{\sigma}$ is the determinant of the co-dimension two metric, ξ^j is the killing vector connected to the conserved charge and $\pi^{[0]i}_j$ is the momentum tensor calculated for the metric with its conserved attributes –mass, angular momentum, electric charge, etc.– turned off. Furthermore, the quantity

$$Q^{[0]}_\xi = -\frac{1}{16\pi G}\int_\Omega d^{D-2}x\sqrt{\sigma}\left(u_i\pi^{[0]i}_j\xi^j\right) \tag{4.2}$$

is introduced with the role of cancelling divergences that come from the asymptotic behaviour of the studied metric.

For the spherically symmetric ansatz 3.13, the killing vector corresponding to the temporal translation invariance is related to the mass of the black hole. Such invariance births from its staticity. Therefore, in Brown and York's scheme, π^t_t is connected to M, similar to what happens with the junction conditions as explained in the above chapter. Indeed, the Brown-York prepotential $\pi^t_t-\pi^{[0]t}_t$ could be considered $[\![\pi^t_t]\!]$ if the inner metric is such that $M_-=0$. However, the comparison works if $\dot{R}$ is sufficiently small, which implies that $\gamma\sim f(r)$. Then, the expressions are worked out employing equations (B.1) and (B.5). This said, the canonical momentum for this particular limit is given by

$$\pi^t_t = \sum_{p=0}^{[\frac{D-1}{2}]}\alpha_p\pi^t_{(p)t} \tag{4.3}$$

where the contribution from the p-th term is given by

$$\pi^t_{(p)t} = p\left(\int_0^1 ds \frac{(D-2)!}{(D-2p-1)!}\frac{f(r)}{r}\left(\frac{1-s^2f^2(r)}{r^2}\right)^{p-1}\right). \tag{4.4}$$

The piece coming from Einstein gravity is obtained simply by studying the $p=1$ case in Eq.(4.4) to get

$$\pi^t_{(1)t} = (D-2)\frac{f(r)}{r}\,. \tag{4.5}$$

With this result, the co-dimension 2 integral (4.1), for a killing vector $\xi^i = \delta^i_t$, gives for the charge the expression

$$Q_t = M\,. \tag{4.6}$$

Noteworthy, the result does not depend on the cosmological constant and the co-dimension 2 integral only introduces a global factor. However, as will be shown in the next lines, it is a fact only for General Relativity.

For the Einstein-Gauss-Bonnet action, the asymptotic expansion of $f^2(r)$ at radial infinity is

$$f^2(r) \approx 1 + \frac{r^2}{\ell^2_{eff}} - \frac{\mu}{r^{D-3}}\left[\frac{1}{1-\frac{2c}{\ell^2_{eff}}(D-3)(D-4)}\right] + \mathcal{O}\left(\frac{1}{r^{2(D-2)}}\right), \tag{4.7}$$

where the effective background spacetime curvature ℓ_{eff} satisfy

$$\frac{1}{\ell^2_{eff}} = \frac{1-\sqrt{1-\frac{4\alpha}{\ell^2}(D-3)(D-4)}}{2\alpha(D-3)(D-4)}\,. \tag{4.8}$$

Consequently, the charge prepotential, which is the same as the above case plus the $p=2$ term gives

$$\pi^t_t = \frac{1}{16\pi G}(D-2)\left[\frac{f(r)}{r} + 2\alpha(D-3)(D-4)\frac{f(r)}{r}\left(\frac{1-\frac{1}{3}f^2(r)}{r^2}\right)\right]. \tag{4.9}$$

After one plugs in this expression in Eq.(4.1) it yields to the conserved charge

$$Q_\xi = M\left[\frac{1-\frac{4}{3}\frac{\alpha}{\ell^2_{\text{eff}}}(D-3)(D-4))}{1-2\frac{\alpha}{\ell^2_{\text{eff}}}(D-3)(D-4))}\right] \tag{4.10}$$

Notice that this expression leads to different conclusions if we are in flat or curved space. In particular, for flat space, it is enough to take the limit $\ell_{\text{eff}} \to \infty$ in the above expression. This action leads to a cancellation of the extra factors and one recovers $Q_\xi = M$. This information was already achieved in Ref.[79]. On the other hand, if the cosmological constant is negative ($\ell^2 > 1$), the r.h.s. of Eq.(4.10) is higher than M, which means that the space dampers the jump. On the contrary, positive cosmological constant ($\ell^2 < 1$) exacerbates it.

Appendix A

Definitions and Conventions in General Relativity

General Relativity and the modified theories of gravity studied in this thesis require some specific notions in mathematics. This chapter aims to define some of these concepts. Einstein summation convention and mostly positive Lorentzian signature is assumed, as well as knowledge in tensor calculus from the reader.

We firstly introduce the metric field $g_{\mu\nu}$, which defines a concept of distance in curved spacetimes. For a manifold parametrized by a set of coordinates $x^\mu = \{t, x, y, z\}$, the line element ds between the points x and $x + dx$ is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,, \tag{A.1}$$

where ds is an infinitesimal distance and $g_{\mu\nu}(x^\sigma)$ encodes the weighting for the different coordinates. For example, the metric for a 4-dimensional Lorentzian flat space in Cartesian coordinates is simply

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2 \,, \tag{A.2}$$

where the metric is the Minkowski one,

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

On the other side, in spherical coordinates $x^\mu = \{t, r, \theta, \phi\}$, the distance is given by

$$ds^2 = -dt^2 + dr^2 + r^2\big(d\theta^2 + \sin^2(\theta) d\phi^2\big) . \tag{A.3}$$

The difference between both descriptions lays in the form and size of an infinitesimal section from the point of view of the particular set of coordinates. In the former, this piece is just a parallelogram, while in the latter it is a spherical shell piece.

From the above rank-two tensor, we have an unique second derivative quantity that happens to transform as a tensor, namely,

$$R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\beta\nu} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\beta\mu} , \tag{A.4}$$

called Riemann or curvature tensor. It is constructed from the Christoffel symbols, which transform as a connection and not as a tensor, whose expression in presence of metricity $\nabla_\mu g_{\alpha\beta}$ for the above metric is

$$\Gamma^\alpha_{\beta\nu} = \frac{1}{2} g^{\alpha\lambda}\big(\partial_\beta g_{\lambda\nu} + \partial_\nu g_{\lambda\beta} - \partial_\lambda g_{\beta\nu}\big) . \tag{A.5}$$

This quantity also allows to define the covariant derivative for a given manifold

$$\nabla_c T^{a_1\cdots a_r}_{b_1\cdots b_s} = \partial_c T^{a_1\cdots a_r}_{b_1\cdots b_s} \tag{A.6}$$

$$+ \Gamma^{a_1}_{dc} T^{da_2\cdots a_r}_{b_1\cdots b_s} + \cdots + \Gamma^{a_r}_{dc} T^{a_1\cdots a_{r-1}d}_{b_1\cdots b_s} \tag{A.7}$$

$$- \Gamma^d_{b_1 c} T^{a_1\cdots a_r}_{db_2\cdots b_s} - \cdots - \Gamma^d{}_{b_s c} T^{a_1\cdots a_r}_{b_1\cdots b_{s-1}d} . \tag{A.8}$$

which, when applied to tensors, maintains covariance.

Between its traces, the Ricci tensor $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$ and Ricci scalar $R = g^{\mu\nu} R_{\mu\nu} = R^\mu_\mu$ appear frequently in the lines below. In addition, of special interest is the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{A.9}$$

that happens to be divergenceless, i.e, $\nabla_\mu G^\mu_\nu = 0$, as a consequence of the Bianchi identity for $R^{\mu\nu}_{\alpha\beta}$.

In order to simplify most of the calculations, we choose Gauss-normal coordinates $x^\mu = (r, x^i)$ suitable for spacetimes with a boundary at fixed r,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = N^2(r)dr^2 + h_{ij}(r,x)dx^i dx^j \,, \tag{A.10}$$

where the spacetime is foliated as an infinite series of spacelike surfaces of constant r, each one with an induced metric h_{ij}. In this frame, we consider the splitting of the spacetime indices $\mu = (r, i)$, such that the different components of the Riemann curvature tensor are given by the Gauss-Codazzi relations in Appendix B. For a foliation with timelike surfaces (corresponding to constant t), it is enough to notice that the change is performed through the change of variables

$$\sqrt{-h} \to i\sqrt{-h} \quad , \quad K_{ij} \to iK_{ij} \quad , \quad N \to iN \,, \tag{A.11}$$

where $i^2 = -1$.

Several quantities in this thesis are written in terms of the totally-antisymmetric Kronecker delta symbol of order p. It is defined as the determinant

$$\delta^{[\nu_1\cdots\nu_p]}_{[\mu_1\cdots\mu_p]} := \begin{vmatrix} \delta^{\nu_1}_{\mu_1} & \delta^{\nu_2}_{\mu_1} & \cdots & \delta^{\nu_p}_{\mu_1} \\ \delta^{\nu_1}_{\mu_2} & \delta^{\nu_2}_{\mu_2} & & \delta^{\nu_p}_{\mu_2} \\ \vdots & & \ddots & \\ \delta^{\nu_1}_{\mu_p} & \delta^{\nu_2}_{\mu_p} & \cdots & \delta^{\nu_p}_{\mu_p} \end{vmatrix} . \tag{A.12}$$

A contraction of $k \leq p$ indices in the Kronecker delta of rank p produces a delta of rank $p-k$,

$$\delta^{[\nu_1\cdots\nu_k\cdots\nu_p]}_{[\mu_1\cdots\mu_k\cdots\mu_p]}\,\delta^{\mu_1}_{\nu_1}\cdots\delta^{\mu_k}_{\nu_k} = \frac{(N-p+k)!}{(N-p)!}\,\delta^{[\nu_{k+1}\cdots\nu_p]}_{[\mu_{k+1}\cdots\mu_p]} \,, \tag{A.13}$$

where N is the range of indices.

Equipped with this knowledge, in the next chapters we can discuss first-order functionals and junction conditions in gravity.

Appendix B

The Gauss-Codazzi-Mainardi relations

For a spacelike foliation (A.10), the Christoffel symbols take the following form

$$\Gamma^r_{rr} = \frac{N'}{N}\,, \qquad \Gamma^r_{ij} = \frac{1}{N}K_{ij}$$
$$\Gamma^i_{jr} = -NK^i_j\,, \qquad \Gamma^i_{jk} = \Gamma^i_{jk}(h)\,. \tag{B.1}$$

where K is the extrinsic curvature

$$K_{ij} = -\frac{1}{2N}\partial_r h_{ij}\,. \tag{B.2}$$

The curvature tensors are given by

$$R^{ri}_{rj} = \frac{1}{N}\left(K^i_j\right)' - K^i_k K^k_j\,, \qquad R^{ri}_{jk} = \frac{1}{N}\left(\bar{\nabla}_j K^i_k - \bar{\nabla}_k K^i_j\right)\,,$$
$$R^{ij}_{rk} = N\left(\bar{\nabla}^i K^j_k - \bar{\nabla}^j K^i_k\right)\,, \qquad R^{ij}_{kl} = \mathcal{R}^{ij}_{kl}(h) - K^i_k K^j_l + K^i_l K^j_k\,. \tag{B.3}$$

Here a prime stands for a radial partial derivative, $\bar{\nabla}$ is the covariant derivative defined with the boundary connection $\Gamma^i_{jk}(h)$ and $\mathcal{R}$ is the boundary Riemann tensor. To rise or lower an index h_{ij} must be used. As a consequence, the Ricci scalar and Ricci tensors can be expressed as

$$R^i_j \;=\; \mathcal{R}^i_j(h) - K^i_j K + \frac{1}{N}(K^i_j)'\,,$$

$$\begin{aligned} R^r_j &= \frac{1}{N}\left(\bar{\nabla}_j K - \bar{\nabla}_k K^k_j\right) \,, \\ R^r_r &= \frac{1}{N}K' - K^i_j K^j_i \,, \\ R &= \mathcal{R}(h) - K^2 - K^i_j K^j_i + \frac{2}{N}K' \,. \end{aligned} \tag{B.4}$$

Specifically, for the metric (3.13) the Christoffel symbols have the form

$$\begin{aligned} \Gamma^r_{rr} &= \frac{-f'(r)}{f(r)} & \Gamma^t_{rt} &= \frac{f'(r)}{f(r)} \\ \Gamma^r_{tt} &= f^3(r)f(r)' & \Gamma^n_{rm} &= \frac{1}{r}\delta^n_m \\ \Gamma^r_{nm} &= -f^2(r)r\,\omega_{nm} & \Gamma^p_{nm} &= \Gamma^p_{nm}(\omega) \end{aligned} \tag{B.5}$$

Another interesting property is the flexibility to define K^{ij}. One way to write it is by simply raise the two indexes with h_{ij} in (B.2). However, considering the identity $h^{ik}h_{kj} = \delta^i_j$ and differentiating we obtain

$$\left(h^{ik}\right)' h_{kj} + h^{ik}\left(h_{kj}\right)' = 0 \,. \tag{B.6}$$

Plugging this in (B.2), the definition become $K^i_j = \frac{1}{2N}h_{jk}\left(h^{ki}\right)'$.

Appendix C

Gauss-normal foliation of Lovelock densities

A Lovelock density is locally equivalent to a boundary term if the dimension of the integration domain is twice the number of Riemann tensors involved. Indeed, the Gauss-Bonnet term in four dimensions or the Lovelock term cubic in the curvature in six dimensions, are examples of such contraction. Therefore, its addition does not modify the equations of motion, fact that can also be seen from Eq.(2.40).

In turn, for $p < \left[\frac{D-1}{2}\right]$, the boundary term lifted to the bulk extra terms which are not identically vanishing. As a consequence, the bulkanization procedure does contribute to the dynamics.

As a starting point, necessary to gain intuition, one may consider the $d-$dimensional integral of β_2, given by

$$\int\limits_{\partial\mathcal{M}} d^dx\,\beta_2 = 2\int\limits_{\partial\mathcal{M}} d^dx\,\sqrt{-h}\,\delta^{[j_1\cdots j_3]}_{[i_1\cdots i_3]}K^{i_1}_{j_1}\left(\mathcal{R}^{i_2i_3}_{j_2j_3} - \frac{2}{3}K^{i_2}_{j_2}K^{i_3}_{j_3}\right) =$$

$$2\int\limits_{\mathcal{M}} d^Dx\,\partial_r\left[\sqrt{-h}\,\delta^{[j_1\cdots j_3]}_{[i_1\cdots i_3]}K^{i_1}_{j_1}\left(\mathcal{R}^{i_2i_3}_{j_2j_3} - \frac{2}{3}K^{i_2}_{j_2}K^{i_3}_{j_3}\right)\right]. \quad \text{(C.1)}$$

The exercise amounts to compute the radial derivative of each tensor in the above equation by Leibniz rule. The derivative acting on the extrinsic curvature can be replaced using (B.3) as

$$\left(K^{i_1}_{j_1}\right)' = N\left(R^{ri_1}_{rj_1} + K^{i_1}_{l}K^{l}_{j_1}\right) . \tag{C.2}$$

On the other hand, one can be worked out an expression which involves the derivative of a determinant, which obeys

$$\left(\sqrt{-h}\right)' = \frac{1}{2\sqrt{-h}}(-h)h^{ij}(h_{ij})' = -K\sqrt{-g}\,. \tag{C.3}$$

The final step in this derivation consists on computing radial derivatives acting on the intrinsic curvature tensor. The problem of this term arises from the fact that the boundary connection Γ^i_{jk} is not a tensor. In variational calculus, the variation of the Riemann tensor can be written as $\delta R^i_{jkl} = \nabla_k \delta\Gamma^i_{lj} - \nabla_l \delta\Gamma^i_{kj}$, where now $\delta\Gamma^i_{kj}$ transforms as a tensor[1]. Fortunately, an expression for the radial derivative of the intrinsic curvature can be derived assuming metricity for the tensor h_{ij}

$$\bar{\nabla}_i h_{jk} = \partial_i h_{jk} - \Gamma^l_{ij}h_{lk} - \Gamma^l_{ik}h_{lj} = 0\,. \tag{C.4}$$

After applying the radial derivative on both sides, one gets the relation

$$\bar{\nabla}_i K_{jk} = \frac{-1}{2N}\left[\left(\Gamma^l_{ij}\right)' h_{lk} + \left(\Gamma^l_{ik}\right)' h_{lj}\right] . \tag{C.5}$$

A precise linear combination of index permutations of the above equation yields the equation

$$\left(\Gamma^k_{ij}\right)' = -N\left(\bar{\nabla}_i K^k_j + \bar{\nabla}_j K^k_i - \bar{\nabla}^k K_{ij}\right) , \tag{C.6}$$

which makes manifest the tensorial character of the radial derivative of the Christoffel symbol. This last fact allows the normal derivative of the intrinsic Riemann tensor to be expressed as

$$\left(\mathcal{R}^i_{jkl}\right)' = \bar{\nabla}_k\left(\Gamma^i_{jl}\right)' - \bar{\nabla}_l\left(\Gamma^i_{jj}\right)' . \tag{C.7}$$

Thus, Eq.(C.1) can be rewritten as

$$\int\limits_{\partial\mathcal{M}} d^d x \beta_2 = 2\int\limits_{\mathcal{M}} d^D x\sqrt{-g}\delta^{[j_1j_2j_3]}_{[i_1i_2i_3]}\Bigg(2K^{i_1}_{j_1}K^{i_2}_{k}R^{ki_3}_{j_2j_3} + 4K^{i_1}_{j_1}K^{i_2}_{j_2}K^{i_3}_{k}K^{k}_{j_3} + 4K^{i_1}_{j_1}\bar{\nabla}_{j_2}\bar{\nabla}^{i_2}K^{i_3}_{j_3}$$

[1]The difference between two connections cancels the non-tensorial part.

$$+\left(R^{ri_1}_{rj_1}+K^{i_1}_{l}K^{l}_{j_1}\right)R^{i_2i_3}_{j_2j_3}-K^{i_1}_{j_1}\left(R^{i_2i_3}_{j_2j_3}+\frac{4}{3}K^{i_2}_{j_2}K^{i_3}_{j_3}\right)K\Bigg).$$

At this point, the identities (D.1) and (D.2) can be recognized, leaving above equation as

$$\int\limits_{\partial\mathcal{M}} d^dx\,\beta_2 = 2\int\limits_{\mathcal{M}} d^Dx\,\sqrt{-g}\delta^{[j_1j_2j_3]}_{[i_1i_2i_3]}\left(R^{ri_1}_{rj_1}R^{i_2i_3}_{j_2j_3}+4K^{i_1}_{j_1}\bar{\nabla}_{j_2}\bar{\nabla}^{i_2}K^{i_3}_{j_3}\right)$$
$$-2\int\limits_{\mathcal{M}} d^Dx\,\sqrt{-g}\delta^{[j_1j_2j_3j_4]}_{[i_1i_2i_3i_4]}K^{i_1}_{j_1}K^{i_2}_{j_2}\left(\frac{1}{2}\mathcal{R}^{i_3i_4}_{j_3j_4}-\frac{1}{3}K^{i_3}_{j_3}K^{i_4}_{j_4}\right). \quad \text{(C.8)}$$

The first term in the above decomposition is one of the contributions needed to construct the Gauss-Bonnet term, whose foliated expression is

$$\int\limits_{\mathcal{M}} d^Dx\mathcal{L}_2 = 2\int\limits_{\mathcal{M}} d^Dx\,\sqrt{-g}\delta^{[i_1\cdots i_3]}_{[j_1\cdots j_3]}\left(R^{rj_1}_{ri_2}R^{j_2j_3}_{i_2i_3}+R^{rj_1}_{i_1i_2}R^{j_2j_3}_{ri_3}\right)$$
$$+\frac{1}{4}\int\limits_{\mathcal{M}} d^Dx\,\sqrt{-g}\delta^{[i_1\cdots i_4]}_{[j_1\cdots j_4]}R^{j_1j_2}_{i_1i_2}R^{j_3j_4}_{i_3i_4}. \quad \text{(C.9)}$$

The second term can be integrated by parts to construct the last missing piece of the foliated Gauss-Bonnet term

$$\delta^{[j_1j_2j_3]}_{[i_1i_2i_3]}R^{ri_1}_{j_1j_2}R^{i_2i_3}_{rj_3} = -4\delta^{[j_1j_2j_3]}_{[i_1i_2i_3]}\bar{\nabla}_{j_2}K^{i_1}_{j_1}\bar{\nabla}^{i_2}K^{i_3}_{j_3}. \quad \text{(C.10)}$$

If one ignores the terms coming from the transversal boundaries, the final relation becomes

$$\int\limits_{\mathcal{M}} d^Dx\mathcal{L}_2 = \int\limits_{\partial\mathcal{M}} d^dx\beta_2 + \int\limits_{\mathcal{M}} d^DxN\mathcal{H}_2$$
$$+4\int\limits_{\mathcal{M}} d^Dx\sqrt{-g}\int_0^1 ds\delta^{[i_1\cdots i_4]}_{[j_1\cdots j_4]}K^{i_1}_{j_1}K^{i_2}_{j_2}\left(\frac{1}{2}\mathcal{R}^{i_3i_4}_{j_3j_4}-s^2K^{i_3}_{j_3}K^{i_4}_{j_4}\right). \quad \text{(C.11)}$$

which splits up the integral of the Gauss-Bonnet term in $D>4$ dimensions as the sum of its corresponding Myers' term and bulk corrections.

Consider now the p-th term of the Lovelock series. Motivated by the previous result in Gauss-Bonnet gravity, we consider the bulkanization of Myers' terms, i.e., taking explicit radial derivative on β_p. Schematically, if one takes $x = \mathcal{R}^{ij}_{kl}$ and $y = K^i_j$, one can rewrite the desired object of study as

$$\begin{aligned}(\beta_p)' = 2p\big(\sqrt{-h}\big)'\sum_{i=0}^{p-1}\binom{p-1}{i}\left(\frac{1}{2i+1}\right)\left(\frac{x}{2}\right)^{p-1-i}(-1)^i y^{2i+1}\\ +px'\sqrt{-h}\sum_{i=0}^{p-2}\binom{p-1}{i}\left(\frac{p-1-i}{2i+1}\right)\left(\frac{x}{2}\right)^{p-2-i}(-1)^i y^{2i+1}\\ +2py'\sqrt{-h}\sum_{i=0}^{p-1}\binom{p-1}{i}\left(\frac{x}{2}\right)^{p-1-i}(-1)^i y^{2i}\,, \end{aligned} \quad \text{(C.12)}$$

where the coefficients coming from parametric integrals were calculated. Replacing equations (C.2), (C.3) and (D.3) and rearranging, the previous equation can be cast in the form

$$\begin{aligned}(\beta_p)' = 2p\sqrt{-g}\int_0^1 ds\delta^{[i_1\cdots i_{2p}]}_{[j_1\cdots j_{2p}]}K^{i_1}_{j_1}K^{i_2}_{j_2}\left(\frac{1}{2}\mathcal{R}^{i_3i_4}_{j_3j_4}-s^2K^{i_3}_{j_3}K^{i_4}_{j_4}\right)\cdots\\ \cdots\left(\frac{1}{2}\mathcal{R}^{i_{2p-1}i_{2p}}_{j_{2p-1}j_{2p}}-s^2K^{i_{2p-1}}_{j_{2p-1}}K^{i_{2p}}_{j_{2p}}\right)\\ +\frac{p}{2^{p-2}}\delta^{[i_1\cdots i_{2p-1}]}_{[j_1\cdots j_{2p-1}]}\left(R^{rj_1}_{ri_2}R^{j_2j_3}_{i_2i_3}+(p-1)R^{rj_1}_{i_1i_2}R^{j_2j_3}_{ri_3}\right)R^{j_4j_5}_{i_4i_5}\times\cdots\times R^{j_{2p-2}j_{2p-1}}_{i_{2p-2}i_{2p-1}}\,. \end{aligned} \quad \text{(C.13)}$$

Notice that the radial foliation of $\mathcal{L}_p$ gives the contributions

$$\begin{aligned}\mathcal{L}_p = \frac{p}{2^{p-2}}\sqrt{-g}\delta^{[i_1\cdots i_{2p-1}]}_{[j_1\cdots j_{2p-1}]}\left(R^{rj_1}_{ri_1}R^{j_2j_3}_{i_2i_3}+(p-1)R^{rj_1}_{i_1i_2}R^{j_2j_3}_{ri_3}\right)R^{j_4j_5}_{i_4i_5}\times\cdots\times R^{j_{2p-2}j_{2p-1}}_{i_{2p-2}i_{2p-1}}+\\ +\frac{1}{2^p}\sqrt{-g}\delta^{[i_1\cdots i_{2p}]}_{[j_1\cdots j_{2p}]}R^{j_1j_2}_{i_1i_2}\times\cdots\times R^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}}\,,\end{aligned}$$

which allow us to identify its part in Eq.(C.13) and write the equality

$$\begin{aligned}\int_{\mathcal{M}} d^Dx\mathcal{L}_p = \int_{\partial\mathcal{M}} d^dx\beta_p + \int_{\mathcal{M}} d^DxN\mathcal{H}_p +\\ +2p\int_{\mathcal{M}} d^Dx\sqrt{-g}\int_0^1 ds\delta^{[i_1\cdots i_{2p}]}_{[j_1\cdots j_{2p}]}K^{i_1}_{j_1}K^{i_2}_{j_2}\left(\frac{1}{2}\mathcal{R}^{i_3i_4}_{j_3j_4}-s^2K^{i_3}_{j_3}K^{i_4}_{j_4}\right)\times\cdots\\ \cdots\times\left(\frac{1}{2}\mathcal{R}^{i_{2p-1}i_{2p}}_{j_{2p-1}j_{2p}}-s^2K^{i_{2p-1}}_{j_{2p-1}}K^{i_{2p}}_{j_{2p}}\right)\,. \end{aligned} \quad \text{(C.14)}$$

This relation shows that, in general, the integral of a Lovelock density in $D > 2p$ dimensions can be decomposed in its Myers' term plus bulk corrections.

Appendix D

Relations between Kronecker deltas of different rank

A recipe to relate Kronecker deltas that differ in rank is needed in several computations. As start point, for the Kronecker delta of rank four, the following set of equivalent contractions

$$\delta^{[j_1\cdots j_4]}_{[i_1\cdots i_4]}K^{i_1}_{j_1}K^{i_2}_{j_2}K^{i_3}_{j_3}K^{i_4}_{j_4} = \delta^{[j_1j_2j_3]}_{[i_1i_2i_3]}\left(KK^{i_1}_{j_1}K^{i_2}_{j_2}K^{i_3}_{j_3} - 3K^{i_1}_{j_1}K^{i_2}_{j_2}K^{i_3}_{l}K^{l}_{j_3}\right) , \tag{D.1}$$

and

$$\begin{aligned}\delta^{[j_1\cdots j_4]}_{[i_1\cdots i_4]}K^{i_1}_{j_1}K^{i_2}_{j_2}\mathcal{R}^{i_3i_4}_{j_3j_4} &= \delta^{[j_1j_2j_3]}_{[i_1i_2i_3]}\left(KK^{i_1}_{j_1}\mathcal{R}^{i_2i_3}_{j_2i_3} - K^{i_1}_{l}K^{l}_{j_1}\mathcal{R}^{i_2i_3}_{j_2j_3} - 2K^{i_1}_{j_1}K^{i_2}_{l}\mathcal{R}^{li_3}_{j_2j_3}\right) ,\\ &= 2\delta^{[j_1j_2j_3]}_{[i_1i_2i_3]}\left(K^{i_1}_{j_1}K^{i_2}_{j_2}\mathcal{R}^{ki_3}_{ki_3} - K^{i_1}_{j_1}K^{i_2}_{k}\mathcal{R}^{ki_3}_{j_2j_3}\right) ,\end{aligned} \tag{D.2}$$

play a fundamental role in the bulkanization method for the Gauss-Bonnet term. Notice that no matter what curvature is involved, the identity still holds. Indeed, this identity holds for any pair of tensors that share the same symmetries with the extrinsic and intrinsic curvature. The generalization of the relations (D.1,D.2) for $2m$ extrinsic curvatures and $n-m$ Riemann tensors is

$$\begin{aligned}&\delta^{[i_1\cdots i_{2n}]}_{[j_1\cdots j_{2n}]}K^{j_1}_{i_1}\cdots K^{j_{2m}}_{i_{2m}}\mathcal{R}^{j_{2m+1}j_{2m+2}}_{i_{2m+1}i_{2m+2}}\cdots\mathcal{R}^{j_{2n-1}j_{2n}}_{i_{2n-1}i_{2n}} =\\ &\quad\delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]}K^{j_1}_{i_1}\cdots K^{j_{2m-2}}_{i_{2m-2}}\mathcal{R}^{j_{2m-1}j_{2m}}_{i_{2m-1}i_{2m}}\cdots\mathcal{R}^{j_{2n-5}j_{2n-4}}_{i_{2n-5}i_{2n-4}}\Big(KK^{j_{2n-3}}_{i_{2n-3}}\mathcal{R}^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}\\ &\qquad-(2m-1)K^{j_{2n-3}}_{l}K^{l}_{i_{2n-3}}\mathcal{R}^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}} - (2m-2j)K^{j_{2n-3}}_{i_{2n-3}}K^{j_{2n-2}}_{l}\mathcal{R}^{lj_{2n-1}}_{i_{2n-2}i_{2n-1}}\Big),\end{aligned} \tag{D.3}$$

where we factored $2m - 2$ extrinsic curvatures and $n - m - 1$ Riemann tensors as common factor.

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